CS 188: Artificial Intelligence

Learning III: Linear regression & Perceptron

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Recap: Decision Tree

- Iterative process, select the most distinguishing/informative attribute to split on next
- Entropy: measure uncertainty in a probability distribution $\langle p_1,\ldots,p_n\rangle$
  $H(\langle p_1,\ldots,p_n\rangle)$ = ______________
  - Quiz: Higher entropy means _____ uncertainty.
    A. more
    B. less

Recap: Decision Tree

- Iterative process, select the most distinguishing/informative attribute to split on next
- Entropy: measure uncertainty in a probability distribution $\langle p_1,\ldots,p_n\rangle$
  $H(\langle p_1,\ldots,p_n\rangle) = \Sigma_i -p_i \log p_i$
- Information gain:
  - reduction on entropy with additional information
- Learning decision tree:
  - Iterative process, selecting the attribute with the highest information gain to split on

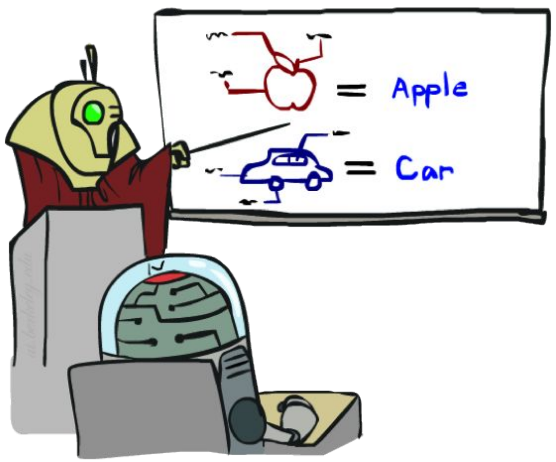
Recap: Model Selection & Hyperparameter Tuning

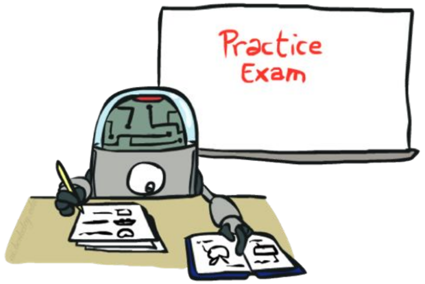
- Data: labeled instances, e.g. emails marked spam/ham
  - Training set
  - Held out set
  - Test set
- Features: attribute-value pairs which characterize each x
- Experimentation cycle
  - Learn parameters (e.g. model probabilities) on training set
  - (Tune hyperparameters on held-out set)
  - Compute accuracy of test set
  - Very important: never "peek" at the test set!
- Evaluation
  - Accuracy: fraction of instances predicted correctly
- Overfitting and generalization
  - Want a classifier which does well on *test* data
  - Overfitting: fitting the training data very closely, but not generalizing well
  - Underfitting: fits the training set poorly

Training Data

Held-Out Data (Validation set)

Test Data

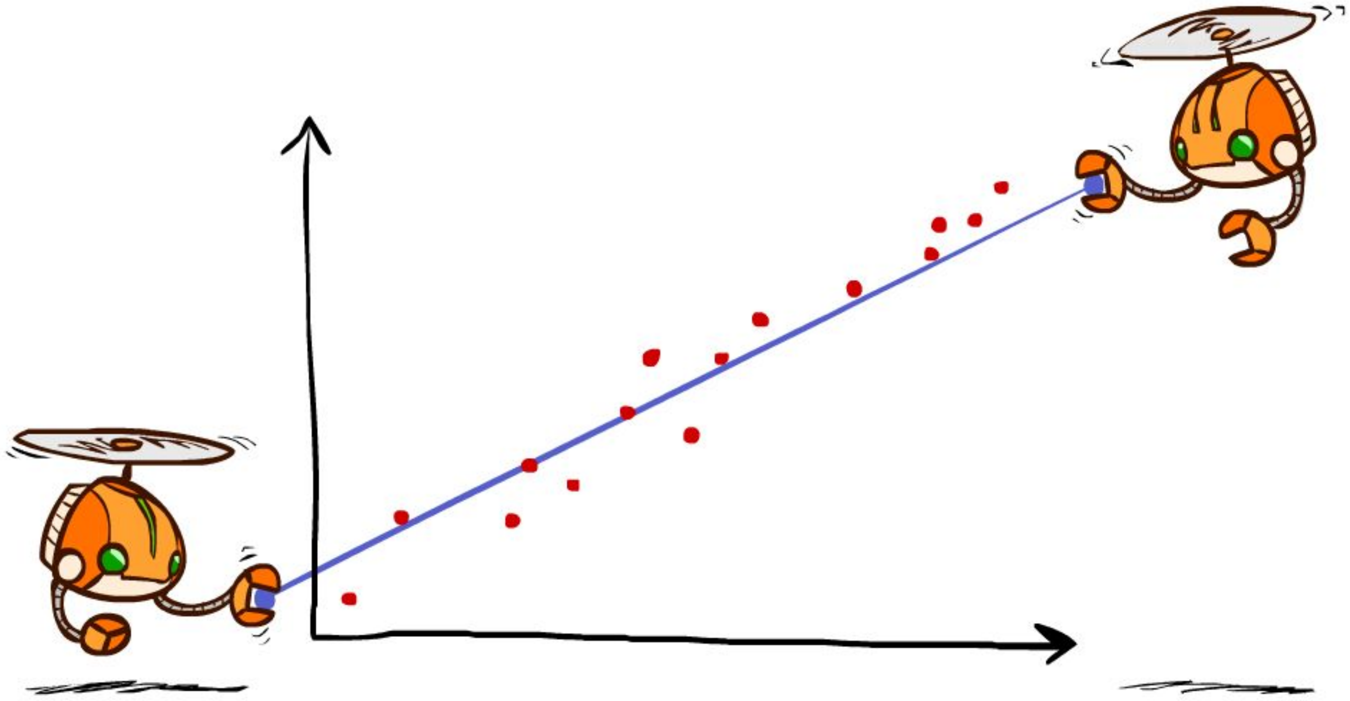




Supervised Learning

- ***Classification*** = learning f with discrete output value
- ***Regression*** = learning f with real-valued output value

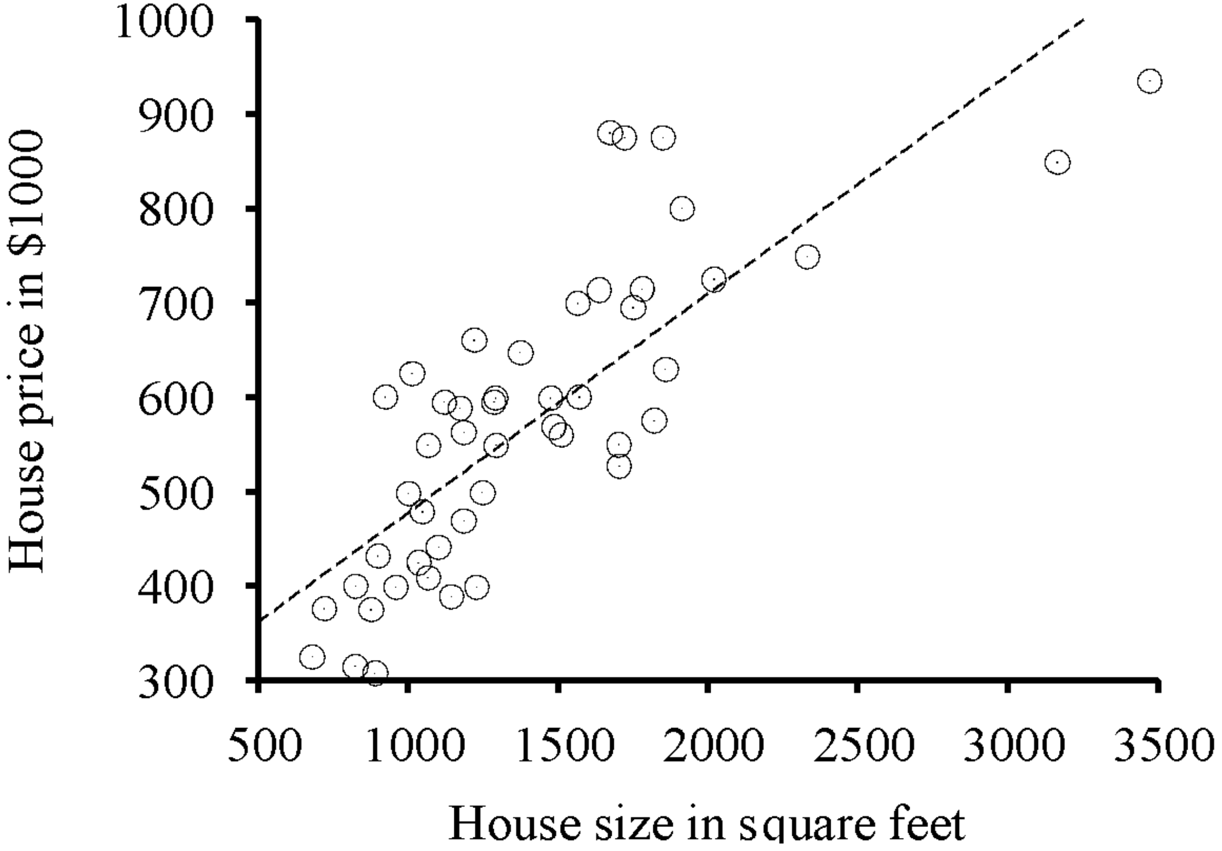
Linear Regression



Hypothesis family: Linear functions

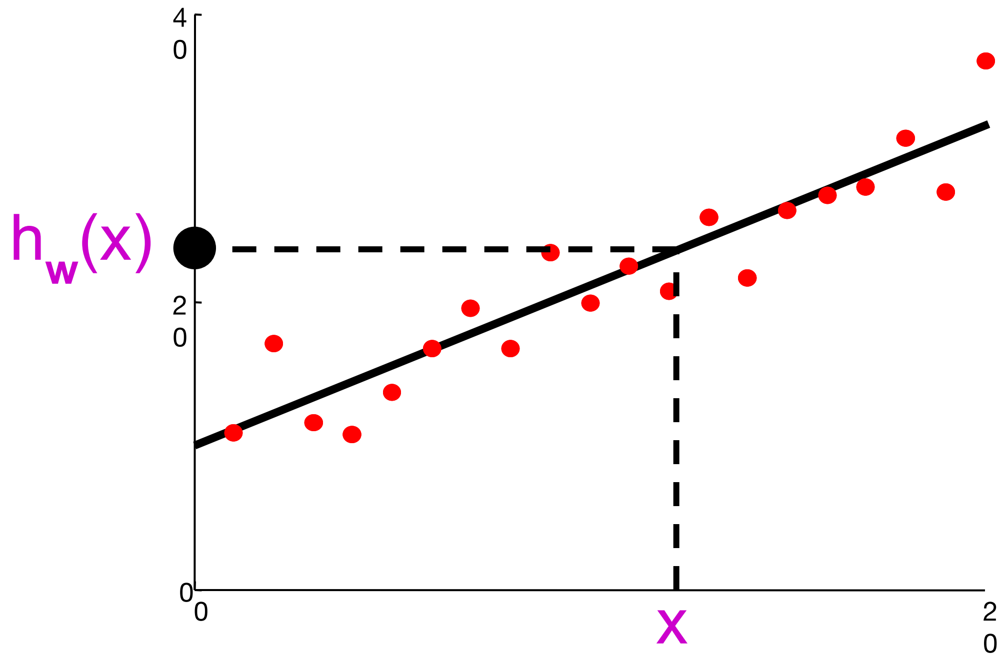
Linear Regression

(x, y=f(x)), x: house size, y: house price

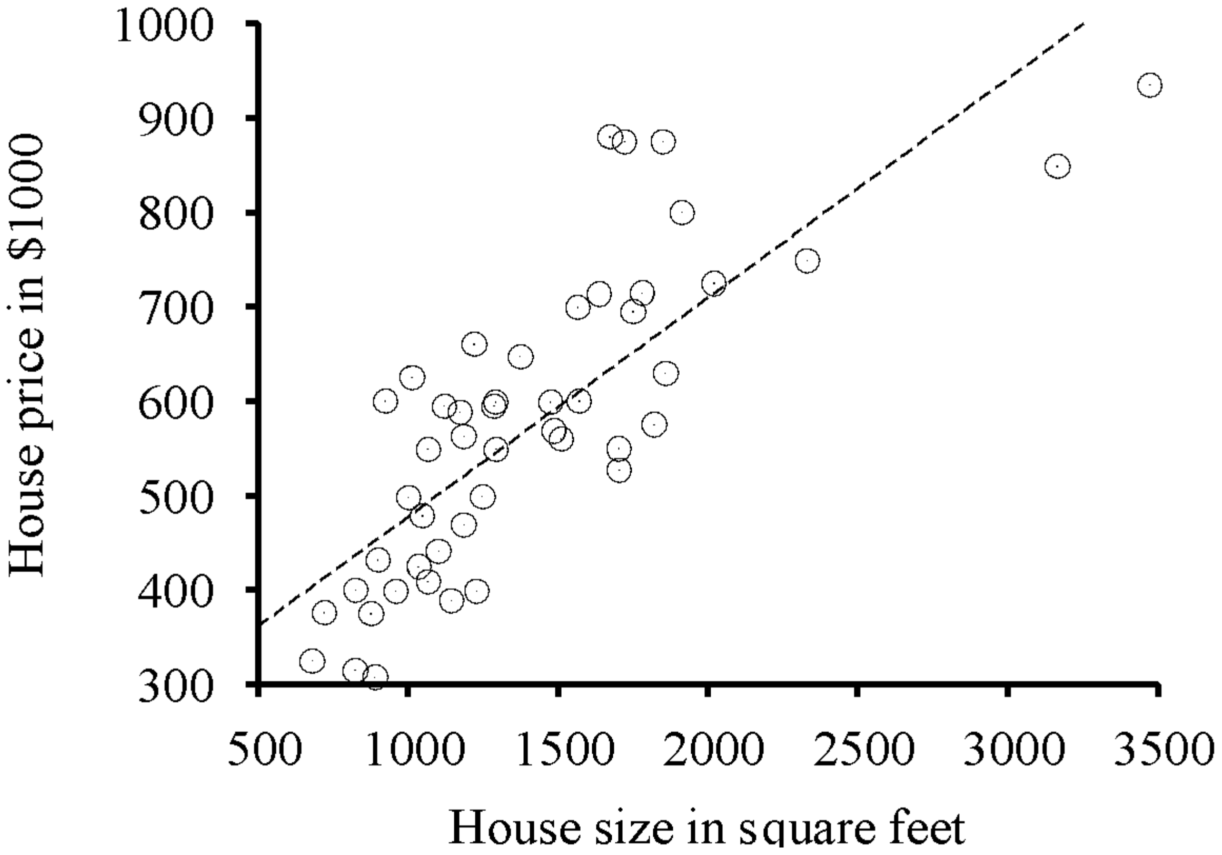


Berkeley house prices, 2009

Linear regression = fitting a straight line/hyperplane



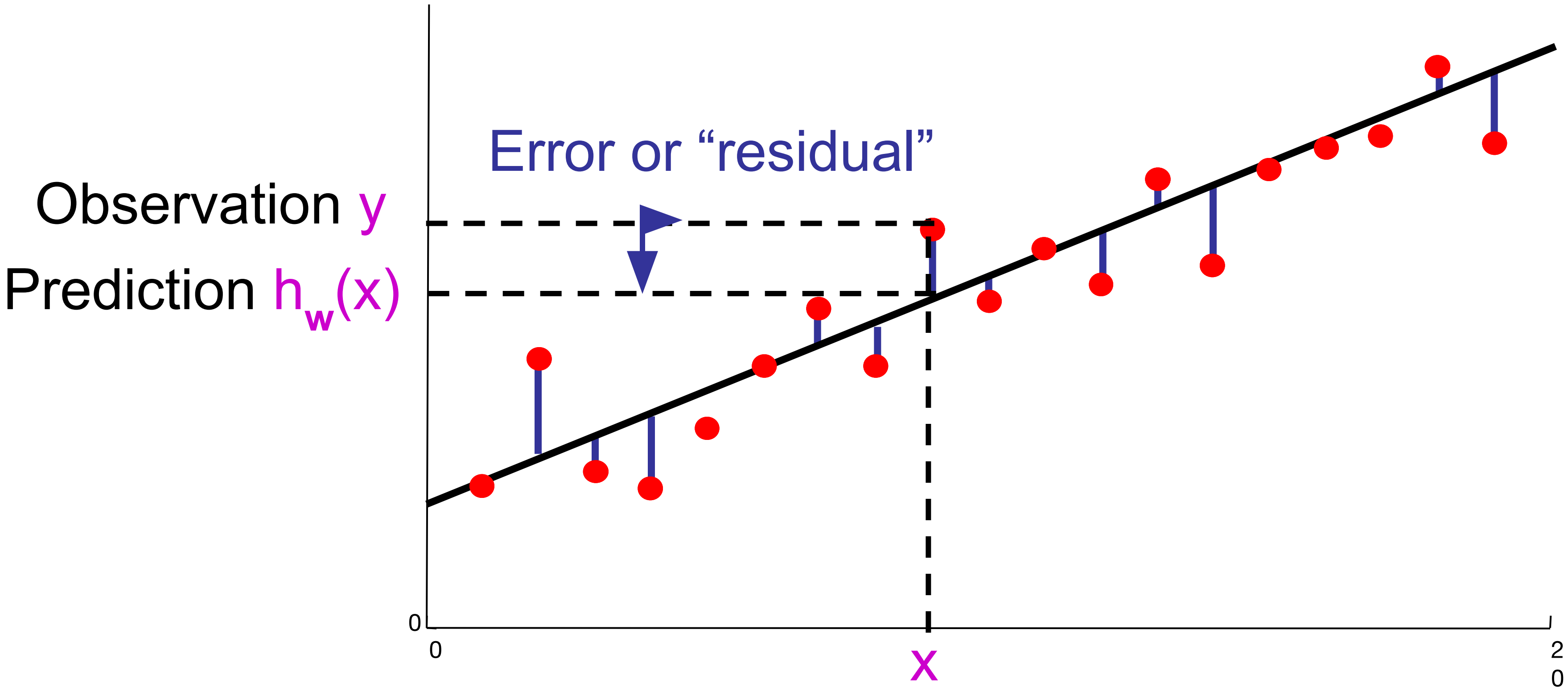
Prediction: $h_w(x) = w_0 + w_1 x$



Berkeley house prices, 2009

Prediction error

Error on one instance: $y - h_w(x)$



Find w

- Define loss function
- Find w* to minimize loss function

Least squares: Minimizing squared error

- L2 loss function: sum of squared errors over all examples
  - Loss = ________________________________
- We want the weights w^* that minimize loss
- At w^* the derivatives of loss w.r.t. each weight are zero:
  - $\partial Loss/\partial w_0$ = ______________________________
  - $\partial Loss/\partial w_1$ = ______________________________
- Exact solutions for N examples:
  - $w_1 = [N\Sigma_j x_j y_j - (\Sigma_j x_j)(\Sigma_j y_j)]/[N\Sigma_j x_j^2 - (\Sigma_j x_j)^2]$ and $w_0 = 1/N\ [\Sigma_j y_j - w_1 \Sigma_j x_j]$
- For the general case where $\mathbf{x}$ is an n-dimensional vector
  - $\mathbf{X}$ is the data matrix (all the data, one example per row); $\mathbf{y}$ is the column of labels
  - $\mathbf{w}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$

Least squares: Minimizing squared error

- L2 loss function: sum of squared errors over all examples
  - Loss = $\Sigma_j (y_j - h_{\mathbf{w}}(x_j))^2 = \Sigma_j (y_j - (w_0 + w_1 x_j))^2$
- We want the weights w^* that minimize loss
- At w^* the derivatives of loss w.r.t. each weight are zero:
  - $\partial\text{Loss}/\partial w_0 = -2 \ \Sigma_j (y_j - (w_0 + w_1 x_j)) = 0$
  - $\partial\text{Loss}/\partial w_1 = -2 \ \Sigma_j (y_j - (w_0 + w_1 x_j)) \, x_j = 0$
- Exact solutions for N examples:
  - $w_1 = [N\Sigma_j x_j y_j - (\Sigma_j x_j)(\Sigma_j y_j)]/[N\Sigma_j x_j^2 - (\Sigma_j x_j)^2]$ and $w_0 = 1/N \, [\Sigma_j y_j - w_1 \Sigma_j x_j]$
- For the general case where x is an n-dimensional vector
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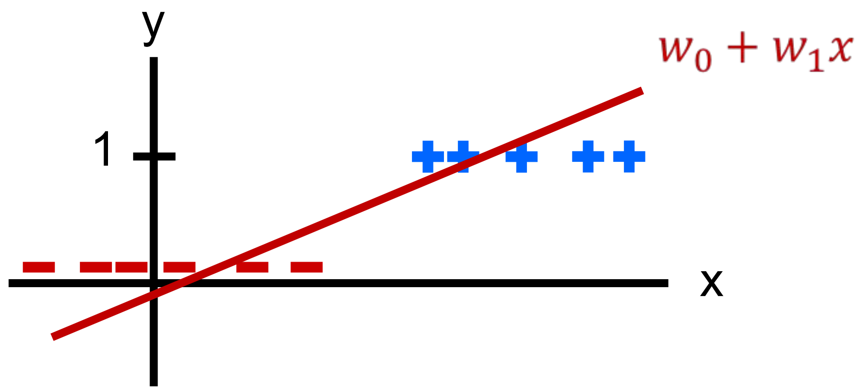
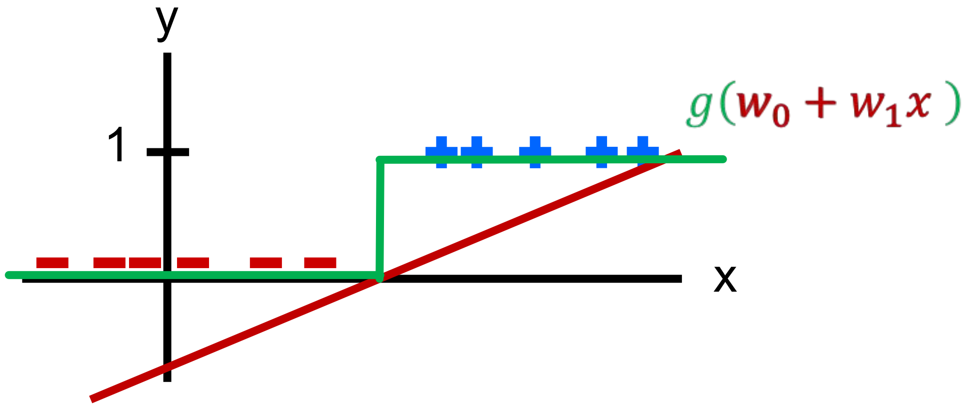
  - $\mathbf{w}^* = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$

Regression vs Classification

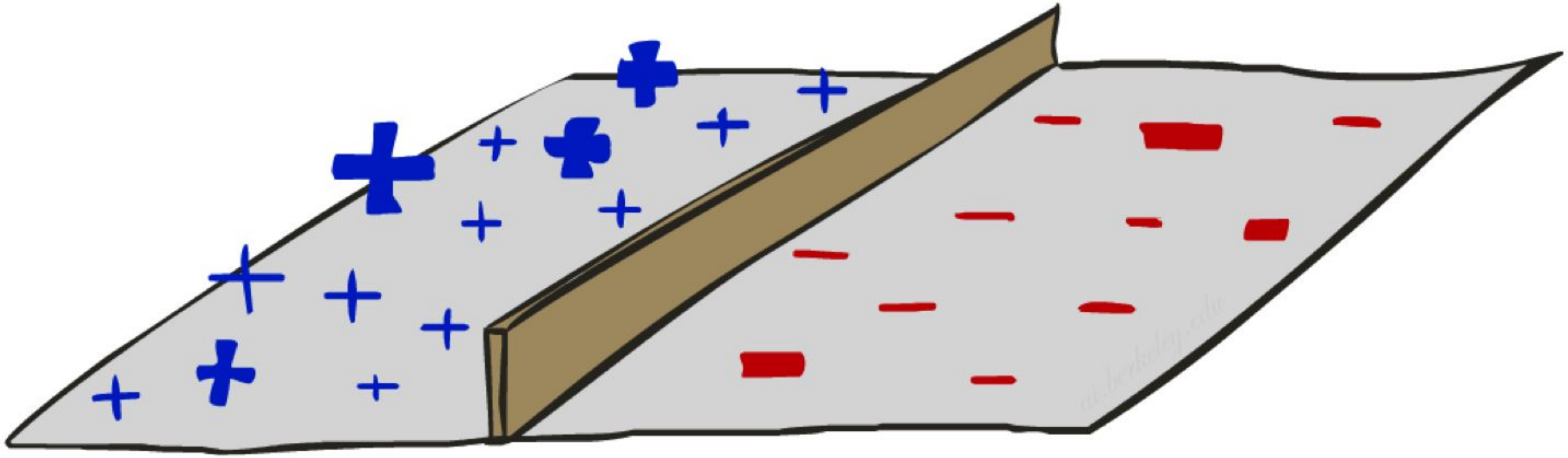
- Linear regression when output is binary, $y \in \{0, 1\}$
  - $h_{\mathbf{w}}(x) = w_0 + w_1 x$



- Linear classification
  - Used with discrete output values
  - Threshold a linear function
  - $h_{\mathbf{w}}(x) = 1,$ if $w_0 + w_1 x \geq 0$
  - $h_{\mathbf{w}}(x) = 0,$ if $w_0 + w_1 x < 0$
  - Activation function g

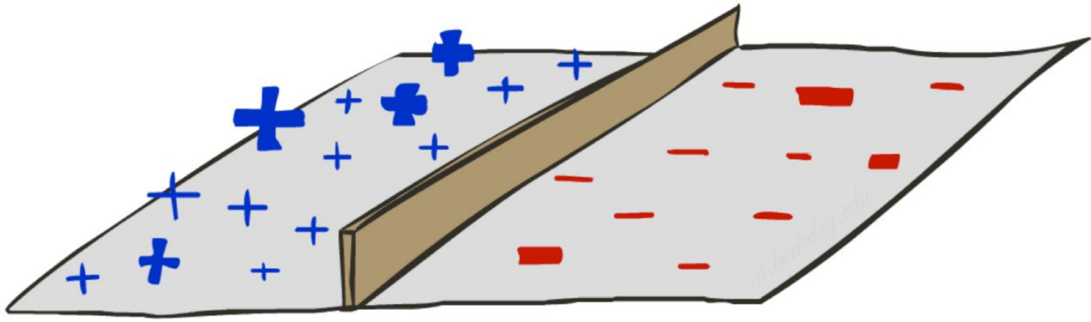


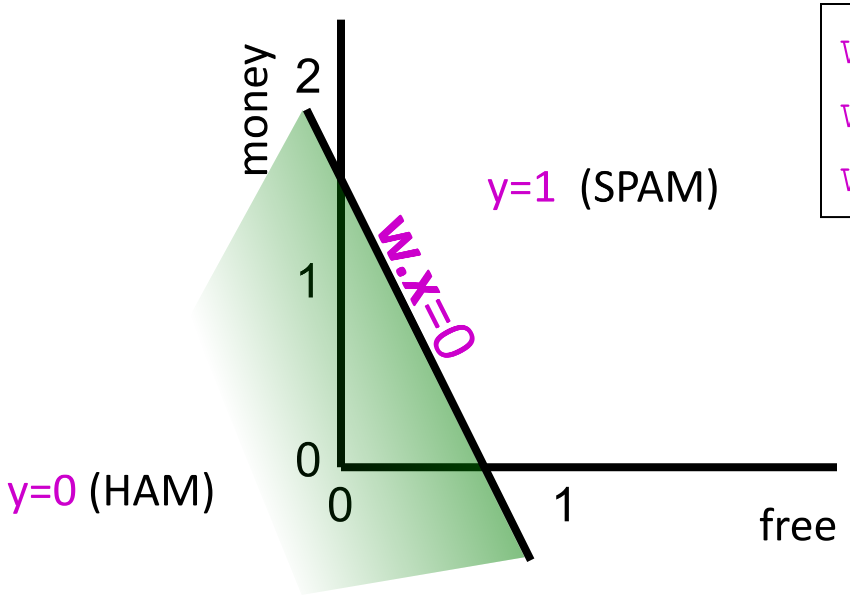
Threshold perceptron as linear classifier



Binary Decision Rule

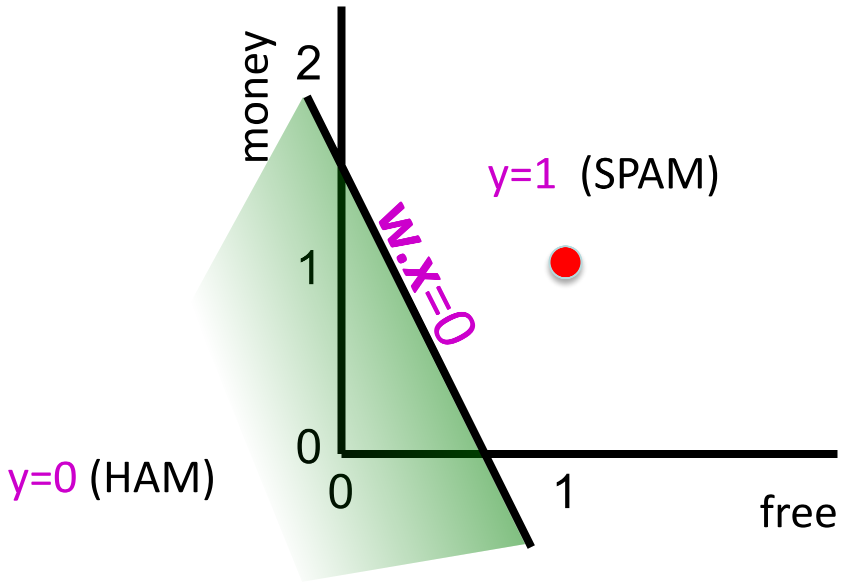
- A ***threshold perceptron*** is a single unit that outputs
  - $y = h_w(\mathbf{x}) = 1$ when $\mathbf{w.x} \geq 0$
    $= 0$ when $\mathbf{w.x} < 0$
- In the input vector space
  - Examples are points $\mathbf{x}$
  - The equation $\mathbf{w.x}=0$ defines a ***hyperplane***
  - One side corresponds to $y=1$
  - Other corresponds to $y=0$





w_0	:	-3
w_{free}	:	4
w_{money}	:	2

Example



w_0	:	-3
w_{free}	:	4
w_{money}	:	2

x_0	:	1
x_{free}	:	1
x_{money}	:	1

Dear Stuart, I'm leaving Macrosoft to return to academia. The **money** is is great here but I prefer to be **free** to do my own research; and I *really* love teaching undergrads!
Do I need to finish
my BA first before applying?
Best wishes
Bill

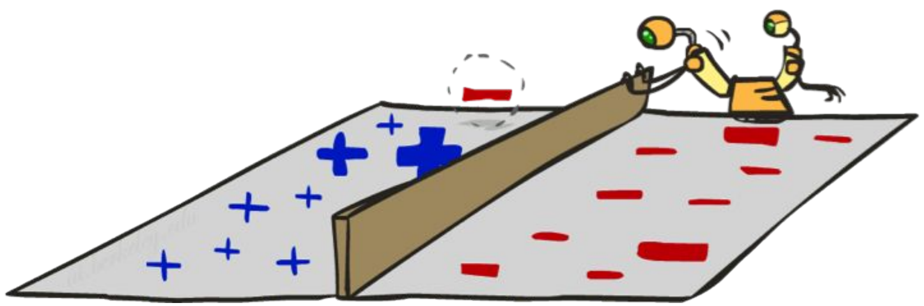
$$\mathbf{w.x} = -3 \times 1 + 4 \times 1 + 2 \times 1 = 3$$

Weight Updates

Need a different solution than before given the characteristic of perceptron

Perceptron learning rule

- If true $y \neq h_{\mathbf{w}}(\mathbf{x})$ (an error), adjust the weights
- If $\mathbf{w.x} < 0$ but the output should be $y=1$
  - This is called a ***false negative***
  - Should ***increase*** weights on ***positive*** inputs
  - Should ***decrease*** weights on ***negative*** inputs
- If $\mathbf{w.x} > 0$ but the output should be $y=0$
  - This is called a ***false positive***
  - Should ***decrease*** weights on ***positive*** inputs
  - Should ***increase*** weights on ***negative*** inputs
- The ***perceptron learning rule*** does this:
  - $\mathbf{w} \leftarrow \mathbf{w} + \underbrace{\alpha}_{\text{learning rate}} \underbrace{(y - h_{\mathbf{w}}(\mathbf{x}))}_{\text{+1, -1, or 0 (no error)}} \mathbf{x}$



Perceptron Learning Rule (Different setup)

- Start with weights = 0
- For each training instance:
  - Classify with current weights

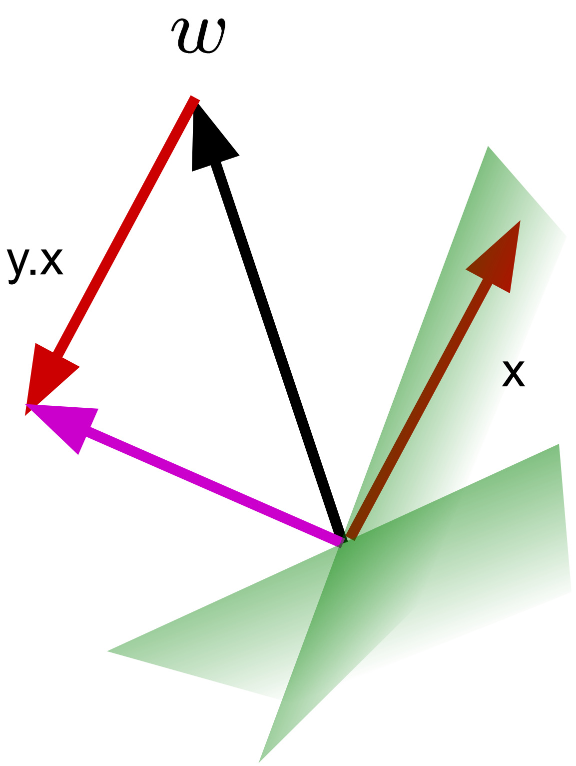
$$y = \begin{cases} +1 & \text{if } w \cdot x \geq 0 \\ -1 & \text{if } w \cdot x < 0 \end{cases}$$

  - If wrong: adjust the weight vector by adding or subtracting the feature vector. Subtract if y is -1.

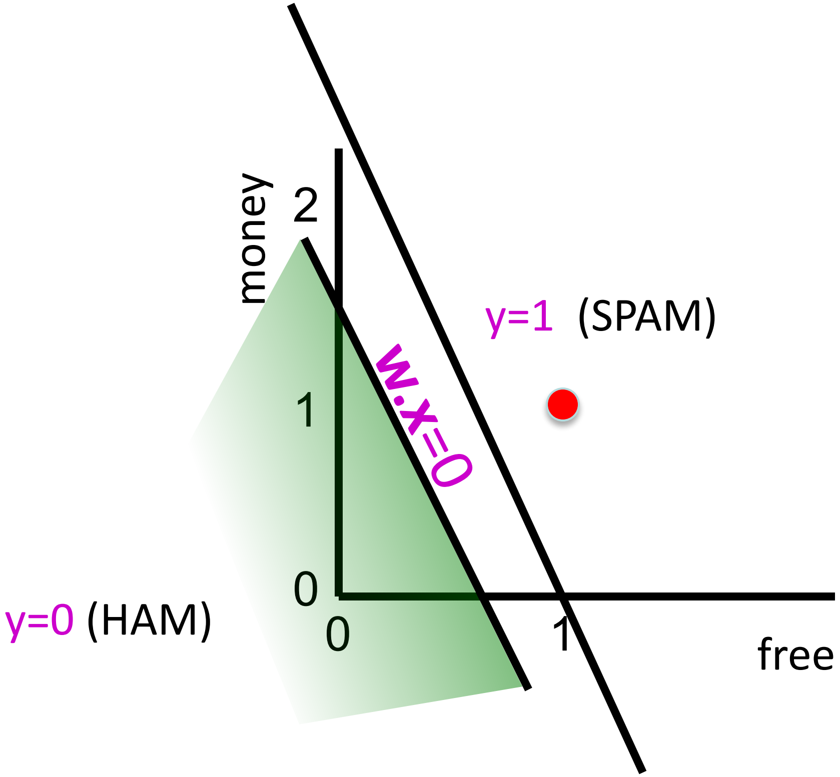
$$w = w + y \cdot \mathrm{x}$$

$y = h_w(\mathbf{x}) = 1$ when $\mathbf{w.x} \geq 0$

$= -1$ when $\mathbf{w.x} < 0$



Example



w_0	: -3
w_{free}	: 4
w_{money}	: 2

x_0	: 1
x_{free}	: 1
x_{money}	: 1

Dear Stuart, I wanted to let you know that I have decided to leave Macrosoft and return to academia. The **money** is is great here but I prefer to be **free** to pursue more interesting research and I *really* love teaching undergraduates! Do I need to finish my BA first before applying?
Best wishes
Bill

$\mathbf{w.x} = -3 \times 1 + 4 \times 1 + 2 \times 1 = 3$

$\mathbf{w} \leftarrow \mathbf{w} + \alpha\,(y - h_{\mathbf{w}}(\mathbf{x}))\,\mathbf{x}$

$\alpha = 0.5$

$\mathbf{w} \leftarrow (-3,4,2) + 0.5\,(0 - 1)\,(1,1,1)$

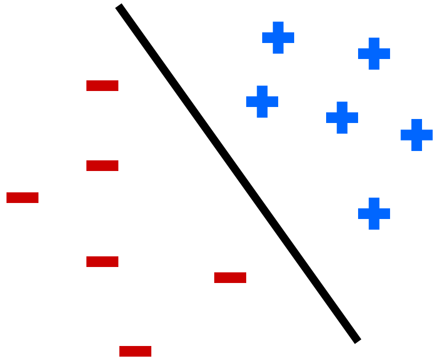
$= (-3.5,3.5,1.5)$

Perceptron convergence theorem

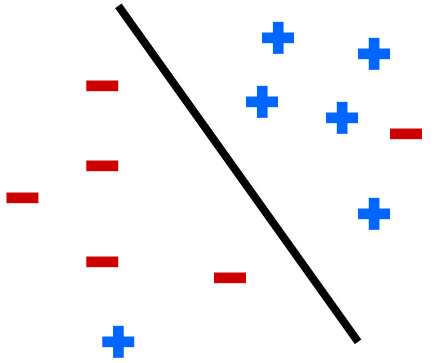
- A learning problem is ***linearly separable*** iff there is some hyperplane exactly separating positive from negative examples

- Convergence: if the training data are ***separable***, perceptron learning applied repeatedly to the training set will eventually converge to a perfect separator

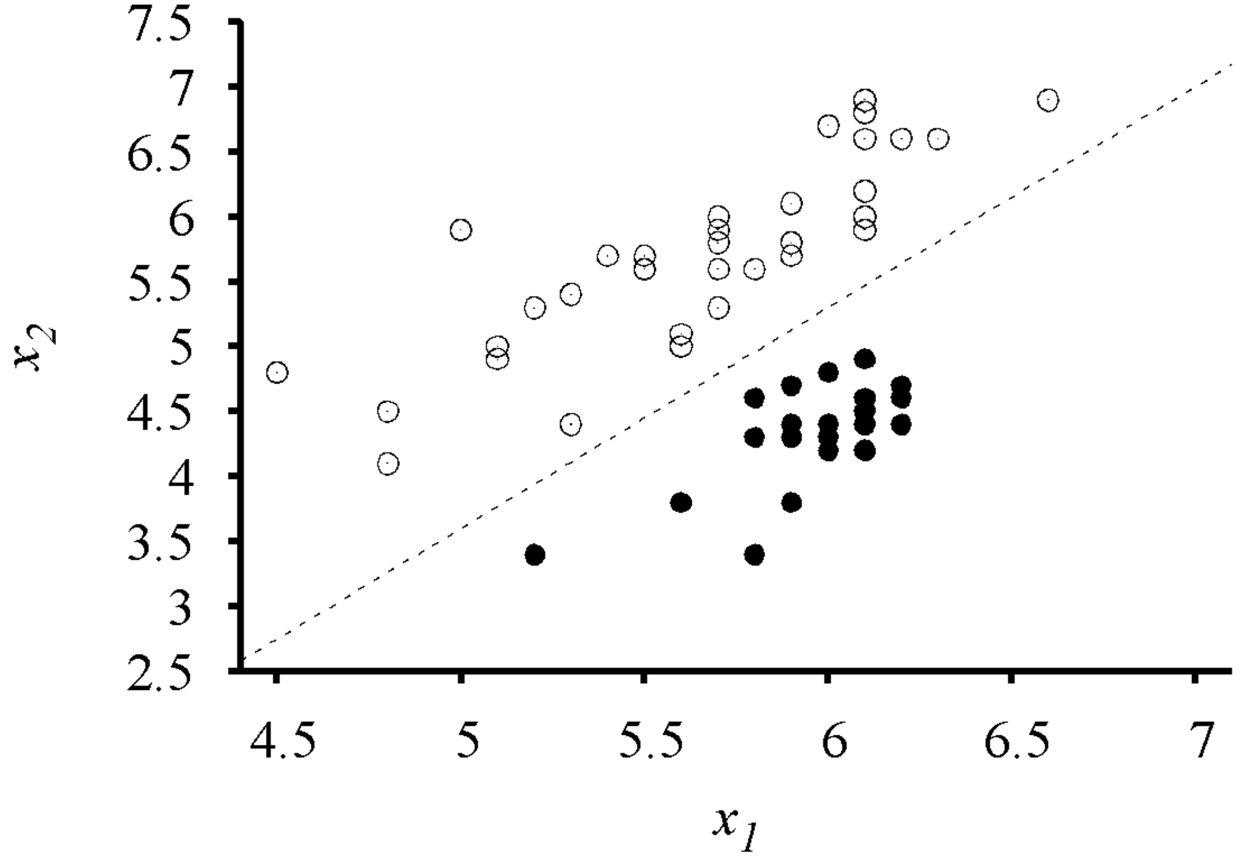
Separable

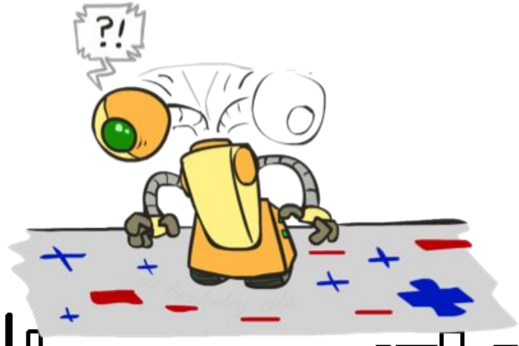


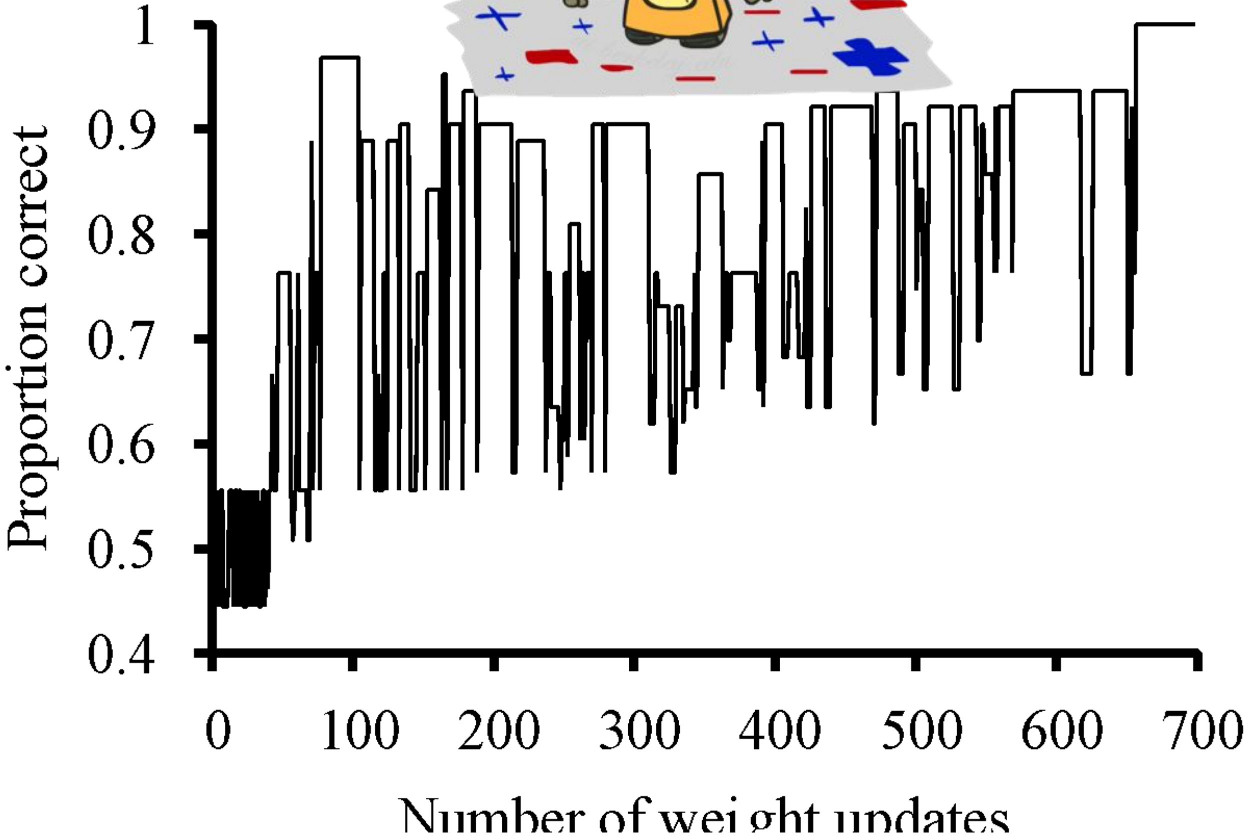
Non-Separable



Example: Earthquakes vs nuclear explosions





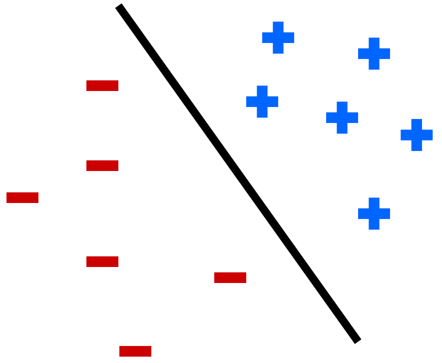


63 examples, 657 updates required

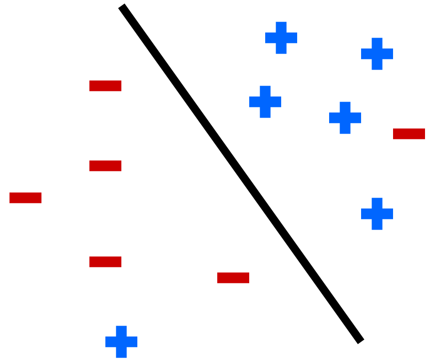
Perceptron convergence theorem

- A learning problem is ***linearly separable*** iff there is some hyperplane exactly separating +ve from –ve examples
- Convergence: if the training data are separable, perceptron learning applied repeatedly to the training set will eventually converge to a perfect separator
- Convergence: if the training data are ***non-separable***, perceptron learning will converge to a minimum-error solution provided the learning rate α is decayed appropriately (e.g., $\alpha=1/t$)

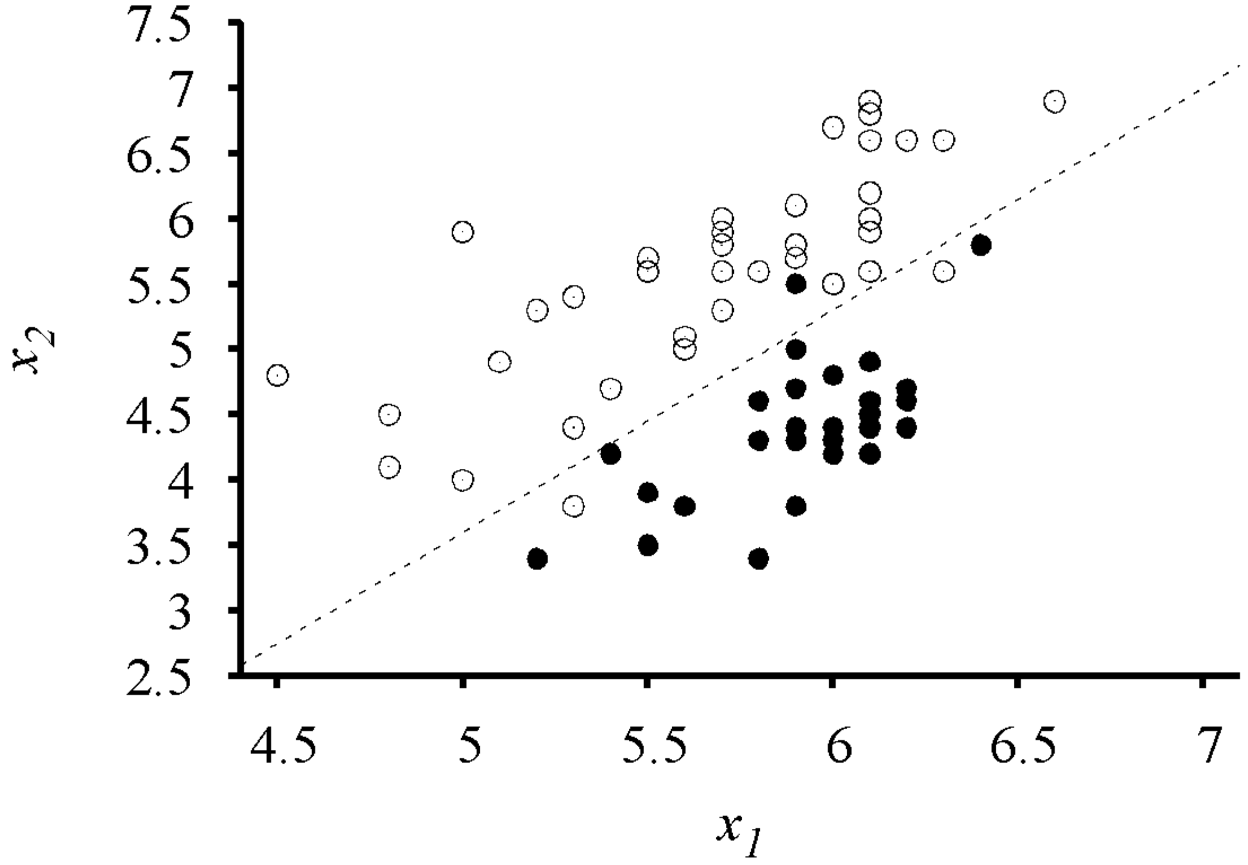
Separable

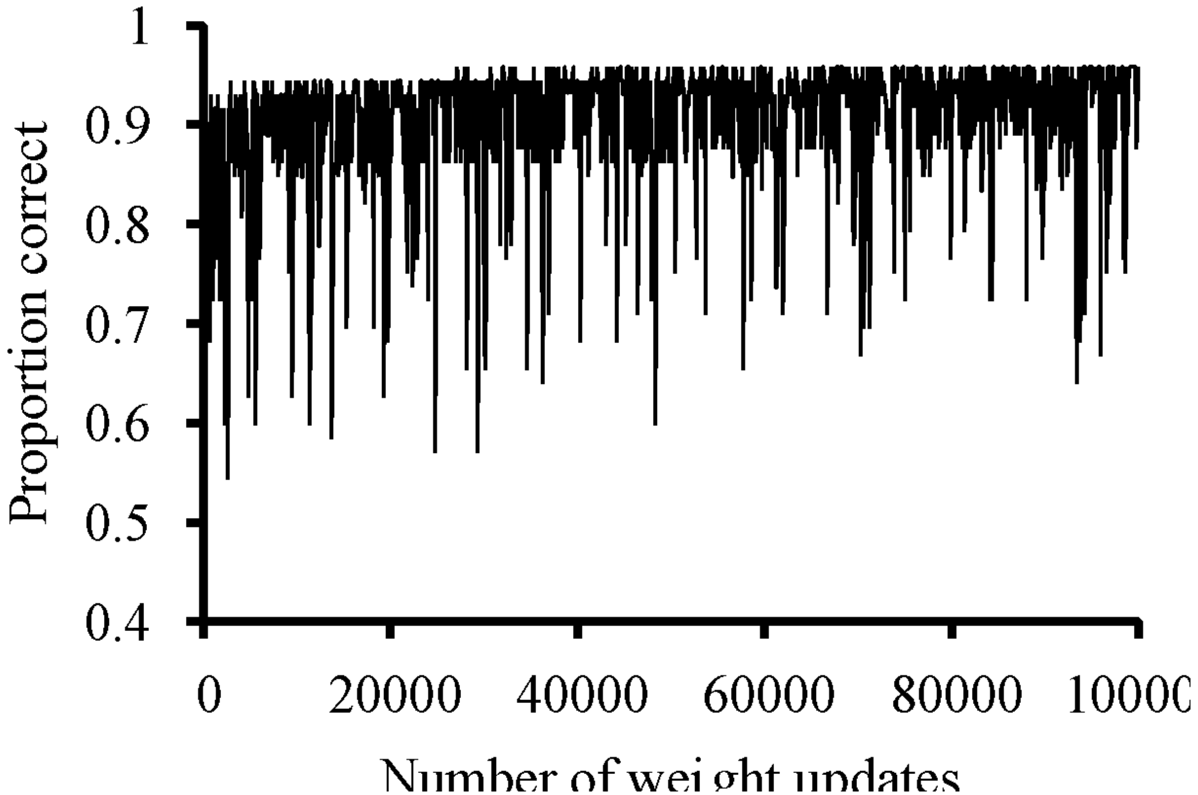


Non-Separable



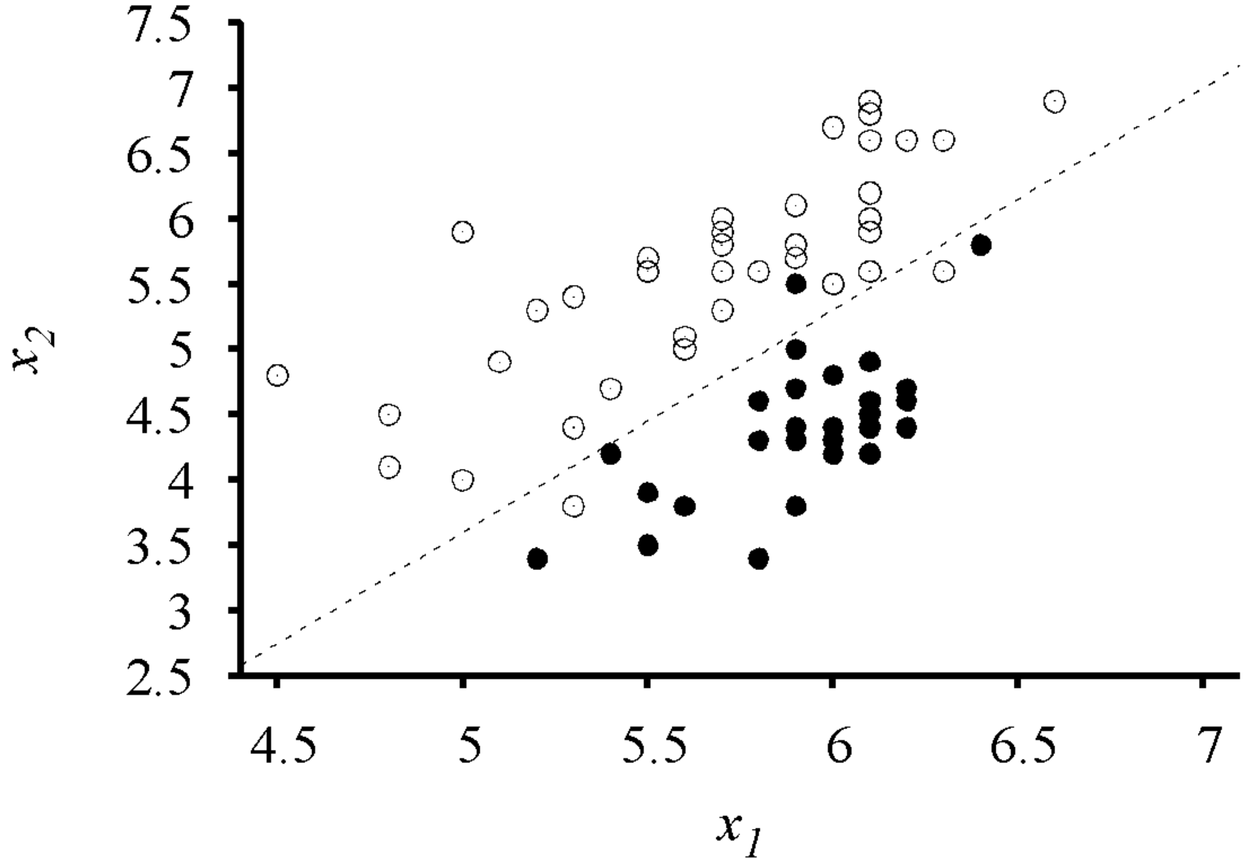
Perceptron learning with fixed α

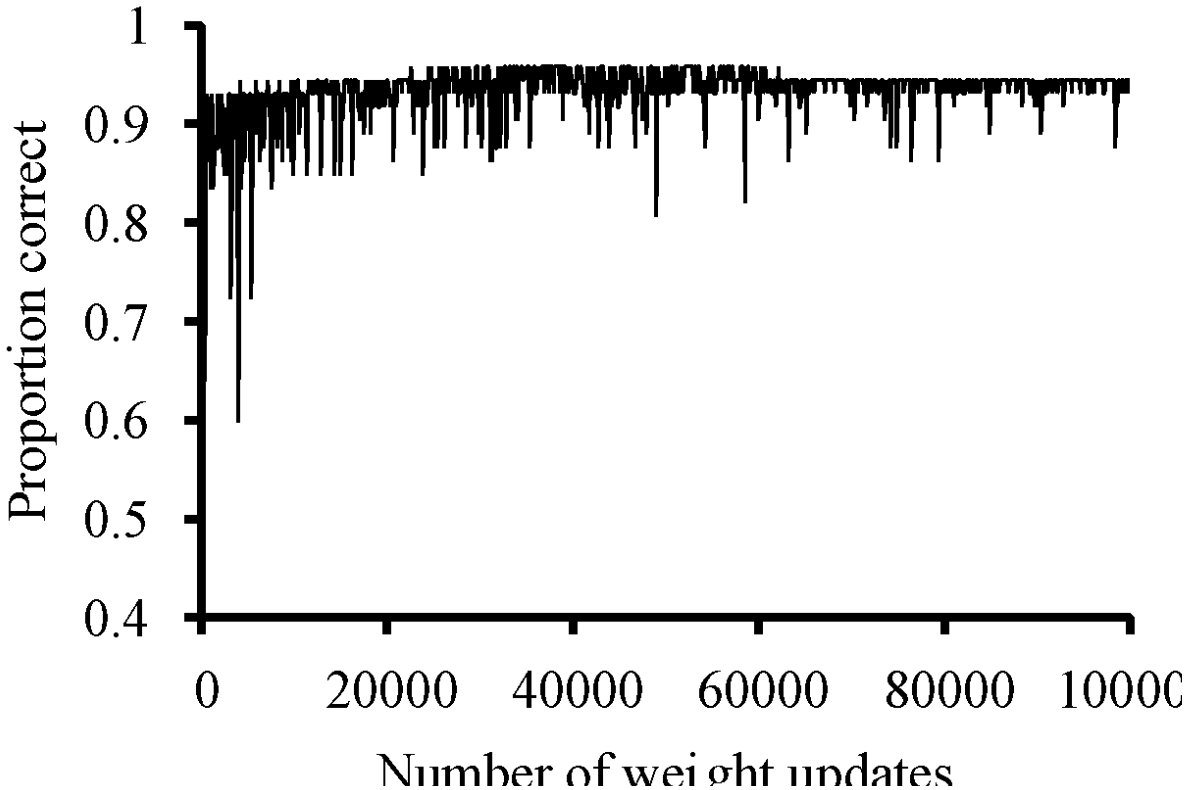




71 examples, 100,000 updates
fixed $\alpha = 0.2$, no convergence

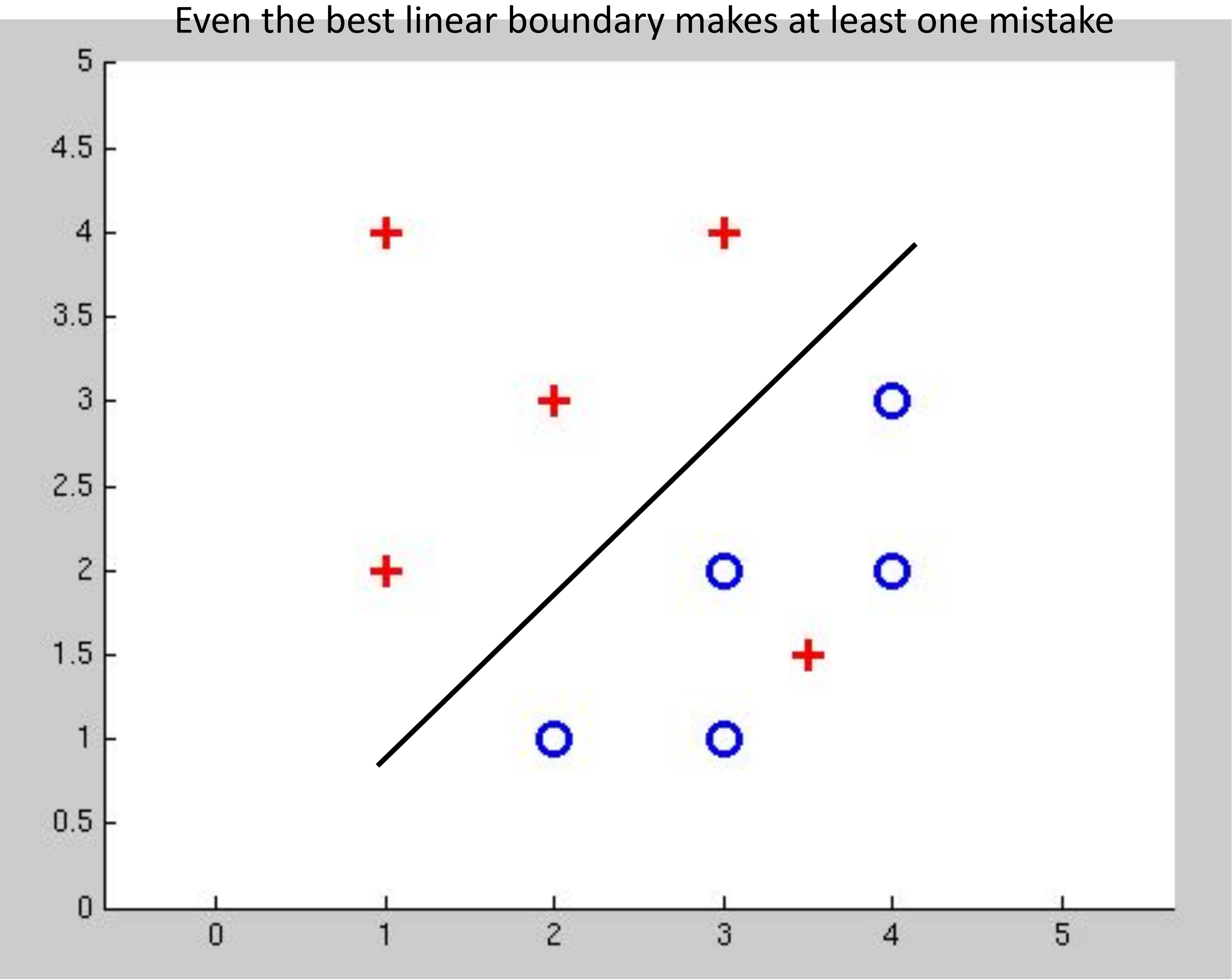
Perceptron learning with decaying α





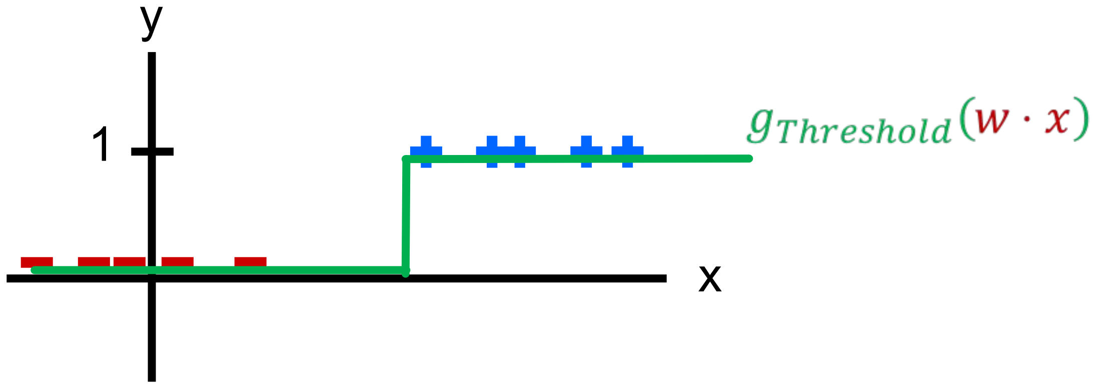
71 examples, 100,000 updates
decaying $\alpha = 1000/(1000 + t)$, near-convergence

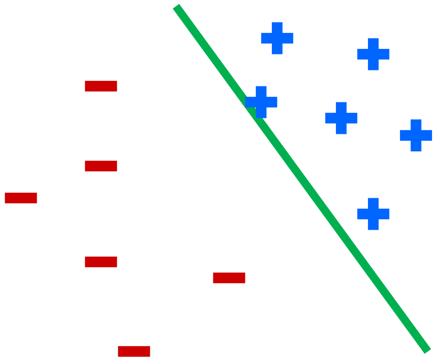
Non-Separable Case

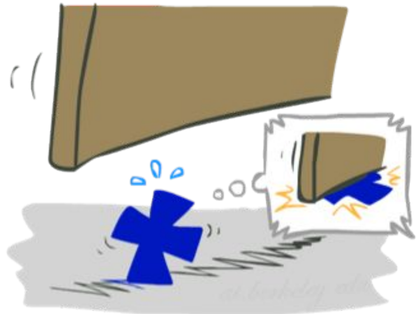


Other Linear Classifiers

- Perceptron is perfectly happy as long as it separates the training data

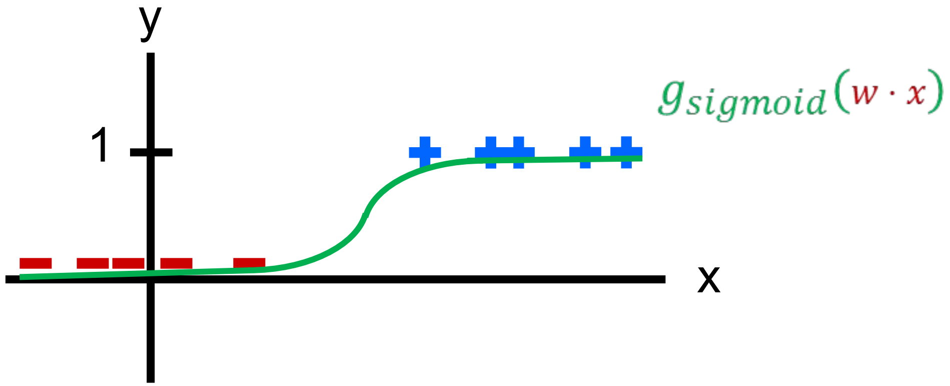




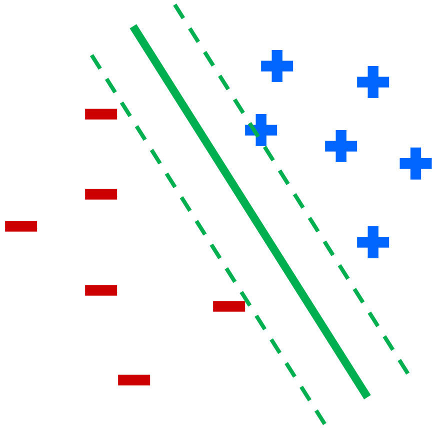


- Logistic Regression

$$g_{sigmoid}(x) = \frac{1}{1 + e^{-x}}$$



- Support Vector Machines (SVM)
  - Maximize margin between boundary and nearest points



Logistic Regression

- Sigmoid function

$$\phi(z) = \frac{1}{1 + e^{-z}}$$