Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Basics

Flow. The *capacity* indicates how much flow can be allowed on an edge. Given a directed graph with edge capacity $c(u, v)$ and s, t, a flow is a mapping $f : E \to \mathbb{R}^+$ that satisfies

- Capacity constraint: $f(u, v) \le c(u, v)$, the flow on an edge cannot exceed its capacity.
- Conservation of flows: $f^{\text{in}}(v) = f^{\text{out}}(v)$, flow in equals flow out for any $v \notin \{s, t\}$

Here, we define $f^{\text{in}}(v) = \sum_{u:(u,v)\in E} f(u, v)$ and $f^{\text{out}}(v) = \sum_{u:(v,u)\in E} f(u, v)$. We also define $f(v, u) = -f(u, v)$, and this is called *skew-symmetry.* Note that the total flow in the graph is $\sum_{v:(s,v)\in E} f(s, v) = \sum_{u:(u,t)\in E} f(u, t)$, where s is the source node of the graph and t is the target node.

Residual Graph. Given a flow network (G, s, t, c) and a flow f, the *residual capacity* (w.r.t. flow f) is denoted by $c_f(u, v) = c_{uv} - f_{uv}$. And the *residual network* $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$.

Ford-Fulkerson. Keep pushing along s-t paths in the residual graph and update the residual graph accordingly. Runs in time $O(mF)$.

2 Max-Flow Min Cut Basics

For each of the following, state whether the statement is True or False. If true provide a short proof, if false give a counterexample.

(a) If all edge capacities are distinct, the max flow is unique.

(b) If all edge capacities are distinct, the min cut is unique.

(c) If all edge capacities are increased by an additive constant, the min cut remains unchanged.

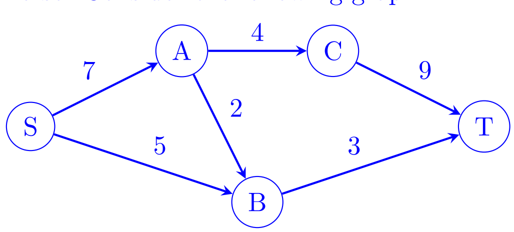
(d) If all edge capacities are multiplied by a positive integer, the min cut remains unchanged.

(e) In any max flow, there is no directed cycle on which every edge carries positive flow.

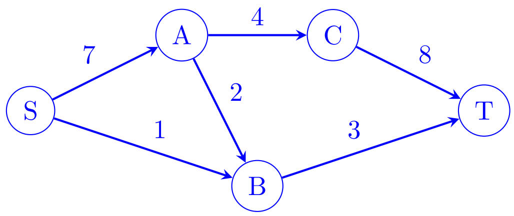
(f) There exists a max flow for which there is no directed cycle on which every edge carries positive flow.

Solution:

(a) False. Consider the following graph:



(b) False. Consider the graph below. The cut SA and the cut SAB are both size 7.

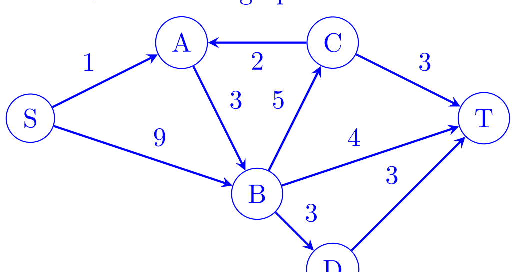


(c) False. Add one to all of the edge capacities of the graph in part (b). The cut SA and the SAB have different values now.

(d) True. Let the value of a cut be $\sum_e c_e$ and the value of the minimum cut be $\sum_{e'} c_{e'}$. The minimum cut must still be the minimum cut after multiplying the edges by a positive constant, due to the distributive property:

$$a\sum_{e'} c_{e'} - a\sum_{e} c_e = a(\sum_{e'} c_{e'} \sum_{e} c_e)$$

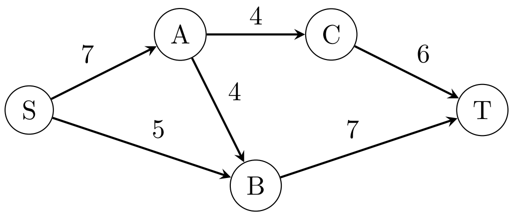
(e) False. Consider the graph below:



(f) True. See any graphs other than (e). Consider the graph in part (e) without the edge SA.

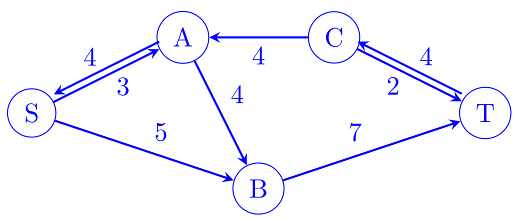
3 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

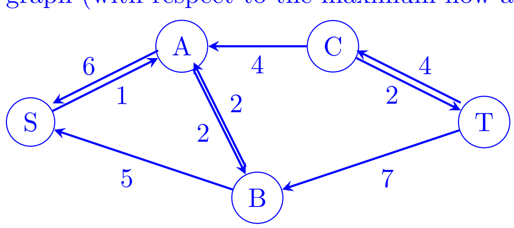


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:



A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

4 Secret Santa (Challenge Problem)

Imagine you are throwing a party and you want to play Secret Santa. Thus you would like to assign to every person at the party another partier to whom they must anonymously give a single gift. However, there are some restrictions on who can give gifts to who: nobody should be assigned to give a gift to themselves or to their spouse. Since you are the host, you know all of these restrictions. Give an efficient algorithm that determines if you and your guests can play Secret Santa.

Solution: Let n be the number of guests. For guest i, make two vertices u_i and v_i. Let $U = \{u_i : i = 1, \ldots, n\}$ and $V = \{v_i : i = 1, \ldots, n\}$. Construct a graph $G = (U \cup V, E)$, where there is an edge between u_i and v_j if guest i can give a gift to guest j. You can play Secret Santa iff G has a perfect matching. Run max-flow on G augmented with a source vertex s connected to every vertex in U and a target vertex t connected to every vertex in V, with edge capacities $c_e = 1$ for every edge, to get a flow f. $\text{size}(f)$ is the size of the largest matching, so G has a perfect matching iff $\text{size}(f) = n$.

5 Taking a Dual

Consider the following linear program:

$$\begin{aligned} \max\ 4x_1 + 7x_2 & \\ x_1 + 2x_2 &\leq 10 \\ 3x_1 + x_2 &\leq 14 \\ 2x_1 + 3x_2 &\leq 11 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Construct the dual of the above linear program.

Solution: If we scale the first constraint by $y_1 \geq 0$, the second by $y_2 \geq 0$, the third by $y_3 \geq 0$, and we add them up, we get an upperbound of $(y_1+3y_2+2y_3)x_1+(2y_1+y_2+3y_3)x_2 \leq (10y_1+14y_2+11y_3)$. We need y_1, y_2, y_3 to be non-negative, otherwise the signs in the inequalities flip. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned}
&\min\ 10y_1 + 14y_2 + 11y_3 \\
y_1 + 3y_2 + 2y_3 &\geq 4 \\
2y_1 + y_2 + 3y_3 &\geq 7 \\
y_1, y_2, y_3 &\geq 0
\end{aligned}$$