Classical bits: 0 or 1

Quantum bits: $|0\rangle$ or $|1\rangle$

"qubits" "ket 0" "ket 1"

or "superposition" of $|0\rangle$ and $|1\rangle$

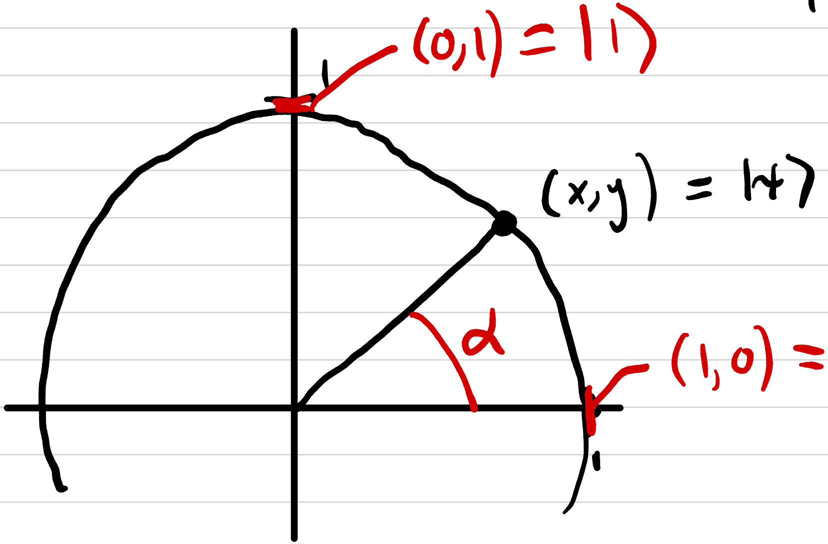
$$|\psi\rangle = x \cdot |0\rangle + y\,|1\rangle$$

$x, y \in \mathbb{R}$ (actually be complex numbers)

$$x^2 + y^2 = 1$$

e.g. $$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$$



$$|\psi\rangle = x \cdot |0\rangle + y \cdot |1\rangle$$
$$= x \cdot (1,0) + y \cdot (0,1)$$
$$= (x, y)$$

quantum states are unit vectors

$$|\psi\rangle = (\cos\alpha) \cdot |0\rangle + (\sin\alpha) \cdot |1\rangle$$

e.g. $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = \cos\frac{\pi}{4} \cdot |0\rangle + \sin\frac{\pi}{4} \cdot |1\rangle$

$$= \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$$

Quantum Measurement

Given $|\psi\rangle = x \cdot |0\rangle + y \cdot |1\rangle$ $\quad (x^2 + y^2 = 1)$

we can "measure" it

w/prob x^2: "observe" outcome 0

$|\psi\rangle$ becomes $|0\rangle$ "collapse of the wave function"

w/prob y^2: "observe" 1

$|\psi\rangle$ becomes $|1\rangle$

e.g. measure $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$: observe 0 w/prob 1/2

observe 1 w/prob 1/2

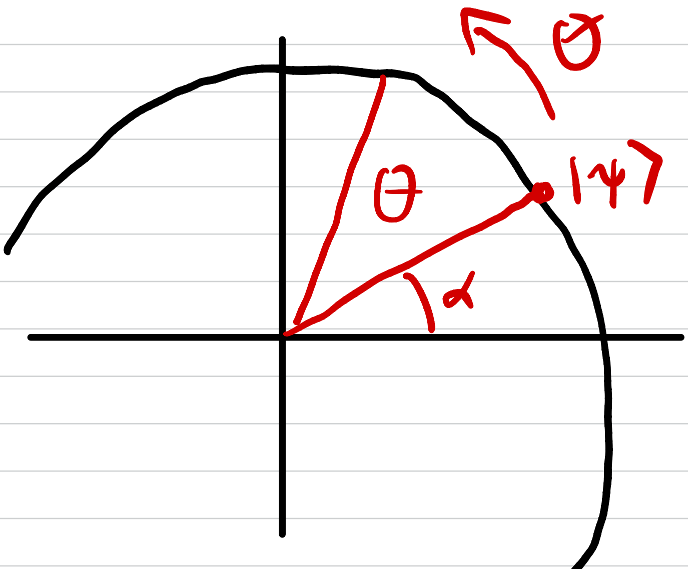
measure $|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$: ditto

measure $|0\rangle$, always observe 0

Quantum operations

(I might tell you about general quantum ops later)

One operation: Rotate$(\theta, |\psi\rangle)$

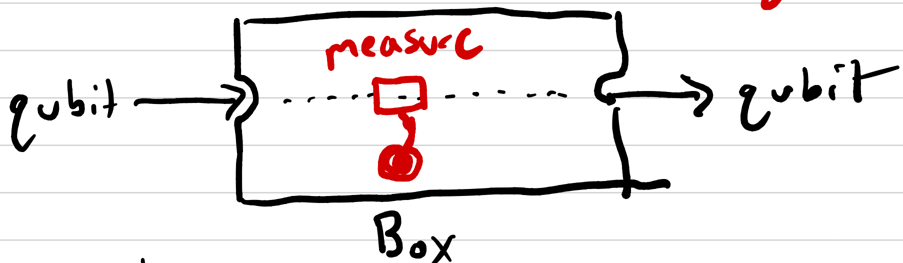


If $|\psi\rangle = \cos(\alpha)|0\rangle + \sin(\alpha)|1\rangle$

moves it to

$\cos(\alpha+\theta)|0\rangle + \sin(\alpha+\theta)|1\rangle$

Elitzur-Vaidman bomb algorithm



Box is either (1) totally empty or (2) contains a bomb

If (1) box is empty and we send $|\psi\rangle = x|0\rangle + y|1\rangle$,
nothing happens, same qubit comes out other end

If (2) bomb, "measures" $|\psi\rangle$

If observes 0, nothing happens
($|0\rangle$ comes out other end)

If observes 1, bomb explodes

Classical strategy

If we send in $|0\rangle$, no info

If we send in $|1\rangle$,

and bomb is in box,

explodes!

Quantum strategy #1

$|\psi\rangle = |0\rangle$
Rotate $(\pi/4, |\psi\rangle)$ $(45°)$
Send $|\psi\rangle$ through box
Rotate $(\pi/4, |\psi\rangle)$

Measure $|\psi\rangle$
If observe 0, output "bomb!"
If observe 1, output "not sure"

If no bomb:

$$|0\rangle \xrightarrow[\pi/4]{\text{rotate}} \cos(\pi/4)|0\rangle + \sin(\pi/4)|1\rangle$$

$$\rightarrow \boxed{\text{empty}} \xrightarrow[\pi/4]{\text{rotate}} \cos\left(\tfrac{\pi}{2}\right)|0\rangle + \sin\left(\tfrac{\pi}{2}\right)|1\rangle$$

$$= 0 \cdot |0\rangle + 1 \cdot |1\rangle = |1\rangle$$

$\therefore$ Alg outputs "not sure" 100% of time

If bomb:

$$|0\rangle \xrightarrow[\pi/4]{\text{rotate}} \cos(\pi/4)|0\rangle + \sin(\pi/4)|1\rangle$$

$$= \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$\rightarrow \boxed{\text{bomb}} \xrightarrow{1/2} \text{output } |x\rangle = |0\rangle$$

$$\boxed{\text{bomb}} \xrightarrow{1/2} \text{observes } 1, \text{ explodes}$$

In first case: $|0\rangle \xrightarrow[\text{by } \pi/4]{\text{rotate}} \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle \xrightarrow{1/2}$ observe 0 "bomb!"

$\xrightarrow{1/2}$ observe 1 "not sure"

w/prob	$1/2$	$1/4$	$1/4$
	explosion	bomb!	not sure

N = Input("Safety level?")

$|\psi\rangle = |0\rangle$

$\theta = (\pi/2)/N$

for $t = 1 \ldots N$

- Rotate$(\theta, |\psi\rangle)$
- Send $|\psi\rangle$ through box

Measure $|\psi\rangle$

- If observe 0, output "Bomb"
- If observe 1, output "Empty"

If empty

$$|0\rangle \xrightarrow[\text{by } \theta]{\text{rotate}} \boxed{\text{empty}} \rightarrow \xrightarrow[\text{by } \theta]{\text{rotate}} \boxed{\text{empty}} \rightarrow$$

$\vdots$

rotate by θ N times

$=$ rotate by $\theta \cdot N = \pi/2$

at end, $|\psi\rangle = |1\rangle$

$\therefore$ Always output "empty"

If contains bomb

$|0\rangle \xrightarrow[\text{by } \theta]{\text{rotate}} \boxed{\text{bomb}} \xrightarrow[\text{measures}]{\text{usually}} 0$ $\quad (|\psi\rangle = |0\rangle)$

$|0\rangle \xrightarrow[\text{by } \theta]{\text{rotate}} \boxed{\text{bomb}} \xrightarrow[\text{measure}]{\text{usually}} 0$ $\quad (|\psi\rangle = |0\rangle)$

$|0\rangle \xrightarrow[\text{by } \theta]{\text{rotate}}$ . . . .

$$\Pr[\text{explosion}] = \Pr[\text{explosion in step 1}] + \Pr[\text{explosion in step 2}] + \dots$$

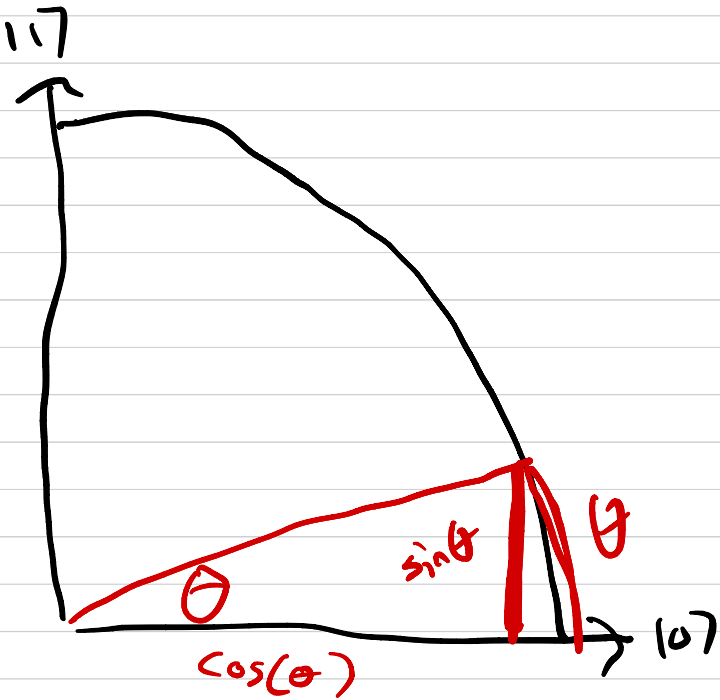
$$\Pr[\text{explosion in step } i] \le \Pr\left[\begin{matrix}\cos(\theta)|0\rangle + \sin(\theta)|1\rangle \\ \text{measures to } 1\end{matrix}\right]$$

$$= \sin^2(\theta)$$

$$\Pr[\text{explosion}] \le N \cdot \sin^2(\theta)$$

$$\Pr[\text{explosion}] \le N \cdot (\sin\theta)^2 \le N \cdot \theta^2$$

$$= N \cdot \left(\frac{\pi/2}{N}\right)^2 = \frac{(\pi/2)^2}{N} \le \frac{2.5}{N}$$



In summary

If empty, 100% of time outputs "empty"

If bomb, explosion w/prob $\leq \frac{2.5}{\sqrt{N}}$

if no explosion output "bomb"