LECTURE 22

Approximation Algorithms

* Vertex Cover ✓
* Vertex Cover via LP ✓
* Metric TSP ✓

Coping with Intractability (NP- hardness)

→ Learn more about input

→ Parameterize differently

→ Heuristics

⋮

→ "Approximation Algorithm"

APPROXIMATION ALGORITHM

Def: For A minimisation problem P, $(\alpha > 1)$

an algorithm is an α- approximation algorithm

if $\forall$ input I from P

$$\text{ALG-OUTPUT}(I) \leq \boxed{\alpha \cdot} \text{OPT}(I)$$

[Maximization problem $(\alpha < 1)$

$$\text{ALGOUTPUT}(I) \geq \alpha \cdot \text{OPT}(I)$$]

MINIMUM VERTEX COVER

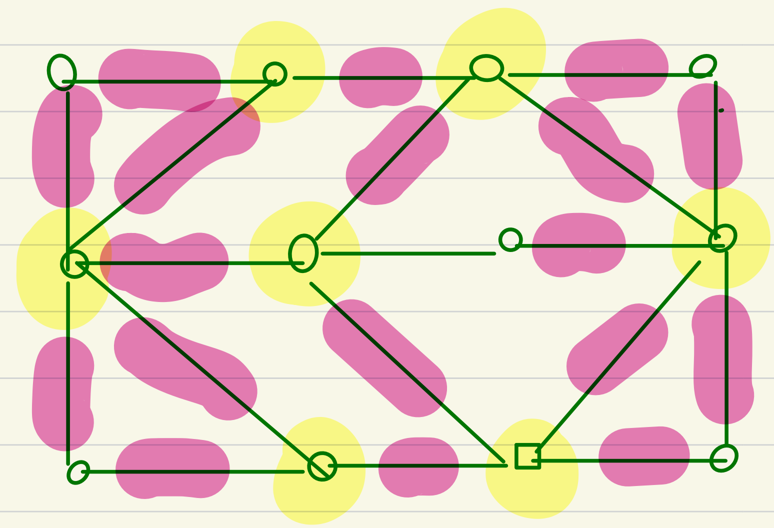
INPUT: Graph $G=(V,E)$

SOL: A vertex cover $S \subseteq V$ of smallest size

S cover all the edges

''

at least one of endpoints $\in S$



$G=(V,E)$

$S = \{$set of vertices$\}$

ALGORITHM

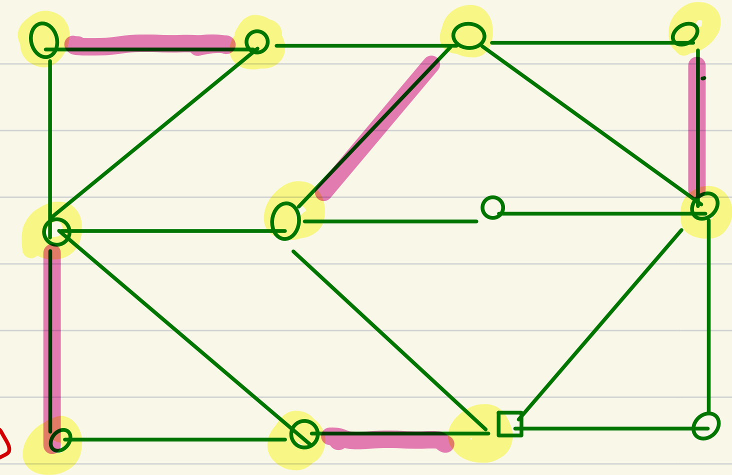
- Pick a maximal matching M

  ||

  matching to which one can add NO more edges

  ||

  Keep picking edges until you can't.



$\boxed{OPT(VC) \geq 5}$

- S = { both endpoints of each edge in M }

  Output $|S| = 2 \cdot |M| \overset{(\text{FACT 1})}{\leq} 2 \cdot |\text{OPTIMAL VERTEX COVER}|$

S is a vertex Cover

Proof: Suppose not. $(u,v) \in E$

(u,v) not covered by S

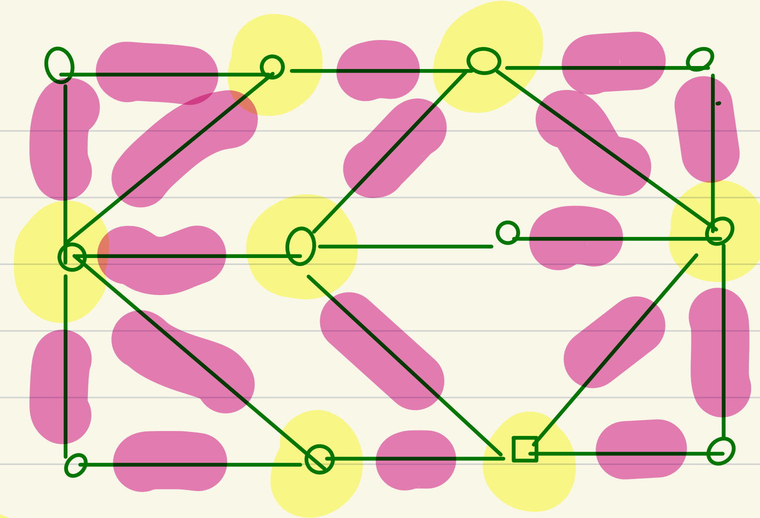
$\Rightarrow$ can add (u,v) to M, M not maximal!

FACT 1 OPTIMAL VERTEX COVER $\geq$ |Maximal Matching M|

Proof: Need 1 vertex to cover each edge of M.

LP ALGORITHM

- Write a LP for vertex Cover



Variable x_i for vertex i

$$x_i = \begin{cases} 1 & \text{if } i \in \text{Optimal Vertex Cover} \\ 0 & \text{otherwise} \end{cases}$$

Constraints:

$$x_i + x_j \geq 1 \quad \forall \; (i, j) \in E$$

$$0 \leq x_i \leq 1$$

Objective:

$$\text{Minimise} \sum_{i=1}^{n} x_i$$

optimal vertex cover is a solution to LP.
But there are also fractional solutions!!

- Solve to get $\{x_i^* ; i = 1 \ldots n\}$ ← optimal solution.

- Claim 0: LP-OPT( ) $\leq$ |OPTIMAL VERTEX COVER|

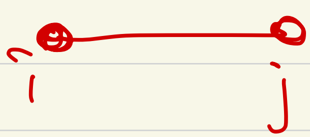
ROUNDING ALGO:

- Optimal LP solution $\{x_i^* \mid i = 1 .. n\}$

- $\begin{cases} x_i^* \geq 1/2 & \rightarrow \text{include in } S \\ x_i^* < 1/2 & \rightarrow \text{do not include in } S \end{cases}$

  Output S.

Claim 1: S is a vertex cover

Proof:  $\rightarrow x_i^* + x_j^* \geq 1 \Rightarrow$ atleast one of $x_i^*, x_j^* \geq 1/2$

$\Downarrow$

$\Leftarrow$ $i \in S$ or $j \in S$

Claim 2: $|S| \leq 2 \cdot \left(\text{LP-OPT} = \sum x_i^* \right)$

Proof: Recall:

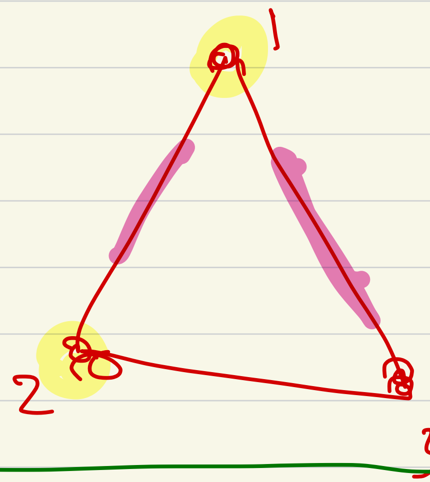
$$\left\{ \begin{array}{ll} x_i^* \geq 1/2 & \\ x_i^* < 1/2 & \end{array} \right\} \begin{array}{l} \longrightarrow \text{include in } S \\ \longrightarrow \text{do not include in } S \end{array}$$

$\forall i \in S \Rightarrow$ Algorithm pays 1 unit

$\Rightarrow$ LP pays $x_i^* \geq 1/2$

$\Rightarrow \underset{\text{of Algorithm}}{\text{Total Cost}} = |S| \leq 2 \cdot \text{LP-OPT}.$

$\leq 2 \cdot |\text{OPTIMAL VERTEX COVER}|$



$$x_1 + x_2 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$x_3 + x_1 \geq 1$$
$$0 \leq x_1, x_2, x_3 \leq 1$$

$\boxed{\text{LP-OPT} = 1/2 + 1/2 + 1/2 = 3/2}$ $\quad x_1 = x_2 = x_3 = 1/2$

OPTIMAL VERTEX COVER $= |\{1, 2\}| = 2$

ALG OUTPUT $= \{1, 2, 3\} = 3$

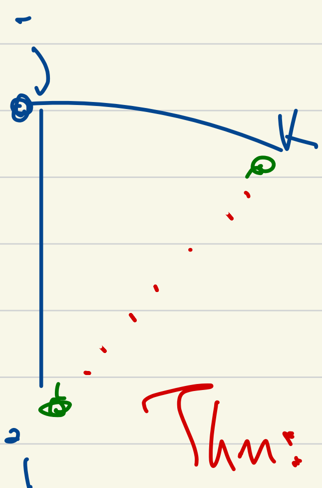
LP-OPT $= 3/2 \leq$ OPTIMAL VERTEX COVER $= 2$

"(METRIC)" TRAVELLING SALESMAN PROBLEM

INPUT: n points with distance $\{d_{ij}\}$ $\forall i, j \in V$

SOL: Minimum cost tour that visits every node exactly once.

METRIC ASSUMPTION: (TRIANGLE INEQUALITY)



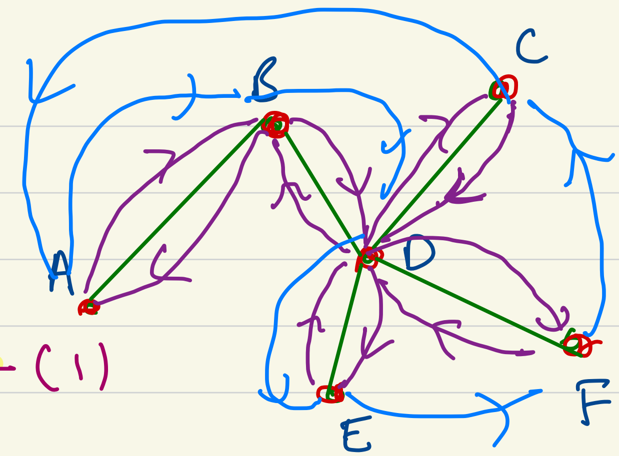
$$d_{ij} + d_{jk} \geq d_{ik} \quad \forall\, i, j, k$$

[Direct routes are shortest]

Thm: GENERAL TSP is hard to approximate to any factor

ALG:

1) Find the MST T



cost (T) ≤ cost (OPTIMAL TSP TOUR) — (1)

2) Depth first traversal of T starting at A

A → B → D → E → D → F → D → C → D → B → A

cost (DFS Traversal) = 2 · Cost (Tree T) — (2)

3) Skip all repeated vertices in the traversal

A → B → D → E → F → C → A (OUTPUT the tour)

Cost (TSP Tour Output) ≤ Cost (DFS Traversal) — (3)

$$\text{Cost}(\text{ALG OUTPUT}) \leq 2 \cdot \text{Cost}\left(\begin{array}{l}\text{OPTIMAL} \\ \text{TSP} \\ \text{TOUR}\end{array}\right)$$