MARCH 31

LECTURE 20

* Reductions Recap
* NP-completeness
* Independent Set $\leq_p$ Integer Programming
* 3SAT $\leq_p$ Independent Set

Problem A $\boxed{\le_P}$ Problem B

1) Reduction Algorithm (converts inputs to A $\to$ inputs to B)

2) $\exists$ A Solution to original input to A $\Longrightarrow$ $\exists$ a solution to input to B

$\exists$ a solution to B $\Longrightarrow$ $\exists$ a solution to A

Problem A is no harder than Problem B.

Remarks: 1) Reduction algorithm needs to run in poly time

$$A \text{ "}\lesssim_{P}\text{"} B$$

2) $A \lesssim_p B$ AND $B \lesssim_p C$

$$\Rightarrow A \lesssim_p C$$

3) A & B ← both are problems with no known algo

NP- Complete Problems

A problem A is NP-complete
if 1) $A \in NP$

2) Every problem $B \in NP$ reduces to A

$$B \leq_p A$$

Corollary: If A & B are NP-complete

$$A \leq_p B \quad \& \quad B \leq_p A$$

Corollary: If $\exists$ a polytime alg for some NP-complete problem

$$\Rightarrow NP = P$$

To show: Problem A is NP-complete

1) $A \in NP$ [Exhibit verification algorithm]

2) Pick some well known NP-complete problem

say $3SAT \leq_p A$

Show that $3SAT \leq_p A$

NP-Complete Problems

"EVERY PROB" ALL OF NP"

$\downarrow$ [Cook70]

CIRCUIT SAT

$\downarrow$ ✓

3SAT

3SAT $\rightarrow$ IND SET

3SAT $\rightarrow$ RUDRATA CYCLE (directed)

IND SET $\rightarrow$ INTEGER PROGRAM

IND SET $\rightarrow$ CLIQUE

IND SET $\rightarrow$ VERTEX COVER

Definition: (Independent Set)

Given a graph $G = (V, E)$, a subset of vertices $S \subseteq V$ is an independent set if there are no edges inside S.

i.e. $\forall\, u, v \in S$, $(u,v) \notin E$.

IND SET:

INPUT: Graph $G=(V,E)$, K

SOL: An ind. set of size K

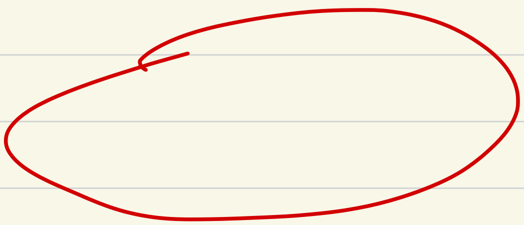
$\leq_P$

INTEGER PROGRAMMING

INPUT: A linear program

SOL: An integer solution to linear program

$G=(V,E)$, K



$\Longrightarrow$

$$x_i = \begin{cases} 1 & \text{if } i \in \text{Independent Set} \\ 0 & \text{otherwise} \end{cases}$$

1) Runs in polytime

2) TO PROVE:

a) G has ind set of size K
$\Longrightarrow$ $\exists$ a solution to IP

IP

$$0 \leq x_i \leq 1$$

$$\sum_{i=1}^{n} x_i = K$$

$$\forall (i,j) \in E \quad x_i + x_j \leq 1$$

b) $\exists$ a solution to IP $\Longrightarrow$ G has an indset of size K

3SAT: fundamental NP-complete

INPUT: 1) Boolean variables $x_1 \dots x_n \in \{0,1\}$

2) Clauses: $\rightarrow (x_1 \vee \bar{x}_2 \vee x_7) \wedge$

$\rightarrow (\bar{x}_5 \vee \bar{x}_6 \vee \bar{x}_8) \wedge$

$\rightarrow (\bar{x}_1 \vee \bar{x}_2 \vee \bar{x}_3) \wedge$

$\vdots$

$\Bigg\}$ m clauses

SOLUTION: An assignment

$$\{x_1 \dots x_n\} \longrightarrow \{0,1\}$$

that satisfies all the clauses

3SAT

INPUT: 3SAT formula on $x_1, \dots, x_n$

$(x_i \vee x_j \vee \bar{x}_k) \wedge (\bar{x}_1 \vee x_2 \vee x_3) \dots$

SOL: A satisfying assignment

$\leq_p$

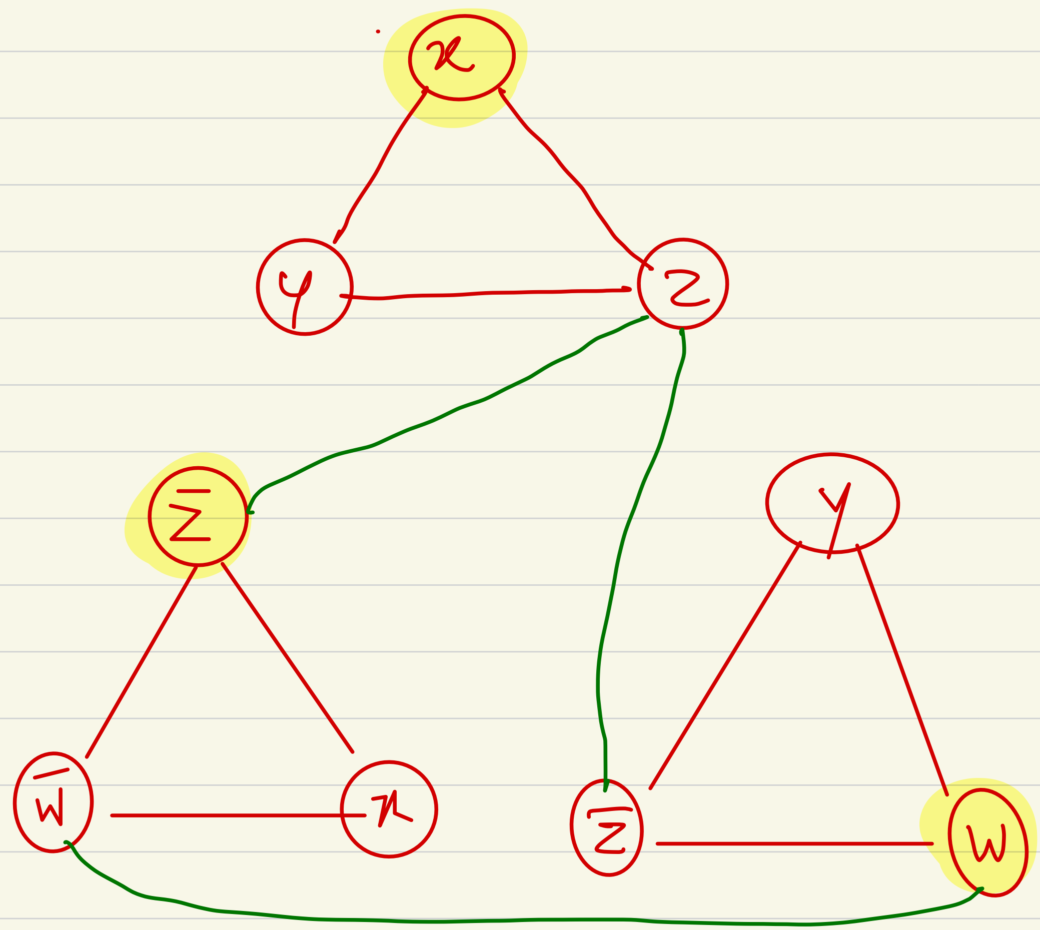
IND SET

INPUT: Graph $G = (V, E)$
integer $\underline{\underline{K}}$

SOL: An independent set of size K

$$(\overset{1}{x} \vee \overset{1}{y} \vee \overset{0}{z}) \wedge$$
$$(\overset{1}{\bar{z}} \vee \overset{1}{\bar{w}} \vee \overset{1}{x}) \wedge$$
$$(\overset{1}{y} \vee \overset{1}{\bar{z}} \vee \overset{0}{w})$$

$\Longrightarrow$



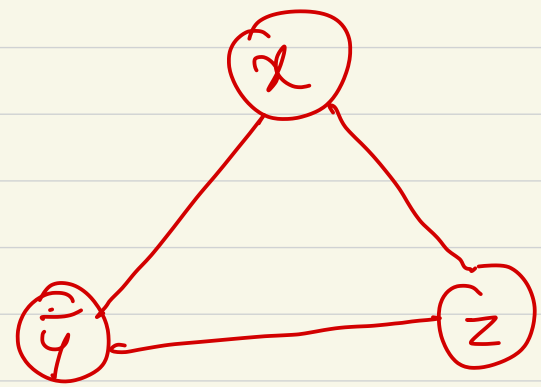
$K = \#$ of clauses

$x = 1$
$y = 1$
$z = 0$
$w = 0$

1) $\forall$ clause $\quad x \vee \bar{y} \vee z$

$\Downarrow$

create a △le



"In the satisfying assignment if $x \vee \bar{y} \vee z$ has true literals $x, \bar{y}$ $\Rightarrow$ add one of those to independent

2) $\forall$ variable x,
add an edge between every vertex labelled x
to
every vertex labelled $\bar{x}$

PROOF:

1) $\exists$ a satisfying assignment

$\Longrightarrow$ $\exists$ a independent set of size k

Proof: If $\{x_1 .. x_n\} \to \{0,1\}$ satisfies the formula then $\forall$ each clause $x_i \vee \bar{x}_j \vee x_k$, pick some true literal, include the vertex in independent set

$\Rightarrow$ $|\text{Independent Set}| = \#$ of clauses

$\Rightarrow$ No edges inside.

∃ an independent set in G with size $K = \#$ triangles $\implies$ ∃ a satisfying assignment

$$\forall \text{ variable } x_i = \begin{cases} 1 & \text{if some vertex } x_i \in \text{Ind Set} \\ 0 & \text{if some vertex } \overline{x_i} \in \text{ind set} \\ \text{arbitrary } 0 & \text{otherwise} \end{cases}$$

Ind Set has x_i then $\overline{x_i} \notin$ Ind Set

$\Rightarrow$ Ind Set picks exactly one vertex in each $\triangle$le

$\implies$ every clause has 1 satisfying literal