APPLICATIONS OF DUALITY

- Max Flow

Flows $\Downarrow$ LP $\xleftrightarrow{\text{duals}}$ Cuts $\Downarrow$ LP^*

Max Flow = Min Cut

(example of strong duality)

- 2 player Zero Sum Game

$\rightarrow$

→ Multiplicative Weights Update (MWU)

→ Application of MWU

→ Prove Strong Duality of LPs

Experts $E_1 \ldots E_n$ (weather forecast)

Setup: On the t^{th} day,

expert E_i incurs a loss $l_i^{(t)} \in \{0, 1\}$

Goal: On the t^{th} day, (randomised) Algorithm A picks an expert E_j, incurs a loss $l_j^{(t)}$

REGRET after T days $=$ (TOTAL LOSS of Algorithm A) $-$ (TOTAL loss of Best Expert in hindsight)

On day t, Algorithm $\mathcal{A}$ outputs

a prob. dist. $x^{(t)} = (x_1^{(t)}, \ldots, x_n^{(t)})$

Loss of Algorithm on Day t $= \sum_{i=1}^{n} x_i^{(t)} \cdot l_i^{(t)}$

Total loss of Algorithm over T days $= \sum_{t=1}^{T} \left[\sum_{i=1}^{n} x_i^{(t)} l_i^{(t)} \right]$

REGRET $= \sum_{t=1}^{T} \left[\sum_{i=1}^{n} x_i^{(t)} l_i^{(t)} \right] - \min_{i=1..n} \left[\sum_{t=1}^{T} l_i^{(t)} \right]$

"Stocks: $S_1 \ . \ . \ . \ S_n$

* Trading stock S_i incurs a loss $\ell_i^{(t)}$ on day t.

* Total capital = 1
* Alg assigns "$x_i^{(t)}$" to stock S_i on day t

$$\text{REGRET} = \sum_{t=1}^{T} \Big[\underbrace{\sum_{i=1}^{n} x_i^{(t)} \cdot \ell_i^{(t)}}_{\text{loss on day } i} \Big] - \min_{i=1..n} \Big[\sum_{t=1}^{T} \ell_i^{(t)} \Big]$$

1) No assumption that the past predicts future performance

?? $\implies$ All algorithms are useless???

2) Experts can even be adversarial!!

STILL $\exists$ an algorithm.

where average regret $\longrightarrow 0$ as. $T \to \infty$

MULTIPLICATIVE WEIGHTS ALG: ($\varepsilon = 0.1$)

- $w_i^{(t)} \leftarrow$ weight of expert i on day t

- $\Pr\left[\text{alg picks expert } i\right] = x_i^{(t)} = \dfrac{w_i^{(t)}}{\sum_{i=1}^{n} w_i^{(t)}}$

- $\{ w_i^{(0)} = 1 \quad \forall i \}$

At end of day t

$$w_i^{(t+1)} = w_i^{(t)} \cdot (1-\varepsilon)^{\ell_i^{(t)}}$$

THM: After T days

Regret of MW algorithm

$$\leq \varepsilon T + \frac{\log_e n}{\varepsilon}$$

Pick $\varepsilon = \sqrt{(\ln n)/T}$

$$\text{REGRET} \leq O\left(\sqrt{\ln n \cdot T}\right)$$

AVG REGRET per day $\leq O\left(\sqrt{\frac{\ln n}{T}}\right)$

PROOF:

Claim: $w_i^{(T)} = (1-\varepsilon)^{\sum_{t=1}^{T} l_i^{(t)}}$

Proof: $w_i^{(T+1)} = w_i^{(T)} \cdot (1-\varepsilon)^{l_i^{(T)}}$

L_t = loss of algorithm on day t

$= \sum_{i=1}^{n} x_i^{(t)} l_i^{(t)}$

Pick any expert E_i

$$\underbrace{(1-\varepsilon)^{\sum_i l_i^{(t)}}} = w_i^{(T)} < \left(\begin{matrix}\text{TOTAL WT}\\ \text{on day } T\end{matrix}\right) = W_T \leq n \cdot \prod_{t=1}^{T} (1-\varepsilon L_t)$$

$$\left(\ln(1-\varepsilon)\right) \cdot \sum_{i=1}^{T} l_i^{(t)} < \ln n + \sum_{t=1}^{T} \ln(1-\varepsilon L_t)$$

$$\Rightarrow \boxed{\sum_{t=1}^{T} L_t - \sum_{t=1}^{T} l_i^{(t)} \leq \varepsilon \sum_{t=1}^{T} l_i^{(t)} + \ln n/\varepsilon \leq \varepsilon T + \frac{\ln n}{\varepsilon}}$$

Claim: $\begin{pmatrix}\text{TOTAL WT} \\ \text{on day } T\end{pmatrix} = W_T \leq n \cdot \prod_{t=1}^{T} (1 - \varepsilon L_t)$

$W_t = \sum \omega_i^{(t)}$

L_t = loss of algorithm on day t

$= \sum_{i=1}^{n} x_i^{(t)} \ell_i^{(t)}$

Proof:

$\boxed{\text{Claim: } W_{t+1} \leq W_t (1 - \varepsilon L_t)}$

Proof:

$$\frac{W_{t+1}}{W_t} = \frac{\sum \omega_i^{(t+1)}}{W_t} = \frac{\sum \omega_i^{(t)} (1-\varepsilon)^{\ell_i^{(t)}}}{W_t}$$

$$= \sum_{i=1}^{n} \left(\frac{\omega_i}{W_t} \right) \cdot (1-\varepsilon)^{\ell_i^{(t)}}$$

$\leftarrow x_i^{(t)}$

$$= \sum_{i=1}^{n} x_i^{(t)} (1-\varepsilon)^{l_i^{(t)}}$$

Fact:

$$\boxed{(1-\varepsilon)^z \leq 1-\varepsilon z}$$

for $\varepsilon \in [0,1]$,

$z \in (0,1)$

$$\leq \sum_{i=1}^{n} x_i^{(t)} (1-\varepsilon\, l_i^{(t)})$$

$$= \left(\sum_{i=1}^{n} x_i^{(t)}\right) - \varepsilon \cdot \left(\sum_{i=1}^{n} x_i^{(t)} \cdot l_i^{(t)}\right)$$

$$= 1 - \varepsilon L_t$$

$\varepsilon = \sqrt{\dfrac{\log n}{T}}$

THM: After T days

Regret of MW

algorithm

$$\leq \varepsilon T + \frac{\log_e n}{\varepsilon} \Bigg\} \leq O\left(\sqrt{\log n \cdot T}\right)$$

Average regret $= O\left(\dfrac{\sqrt{\ln n \, T}}{T}\right) \approx O\left(\dfrac{\sqrt{\log n}}{\sqrt{T}}\right)$

wts on day 0	1	1	1	1
on day 1 loss :	1	0	1	0
wts on day 2 :	$1 \cdot (1-\varepsilon)^1 = 1-\varepsilon$	$1 \cdot (1-\varepsilon)^0 = 1$	$1 \cdot (1-\varepsilon)^1 = 1-\varepsilon$	$1 \cdot (1-\varepsilon)^0 = 1$

ZERO SUM GAME

→ $A[\cdot, \cdot]$

→ Row player picks row r

→ Colm player picks col c

Rowplayer : Payoff = $A[r, c]$

Rowplayer p dist over rows

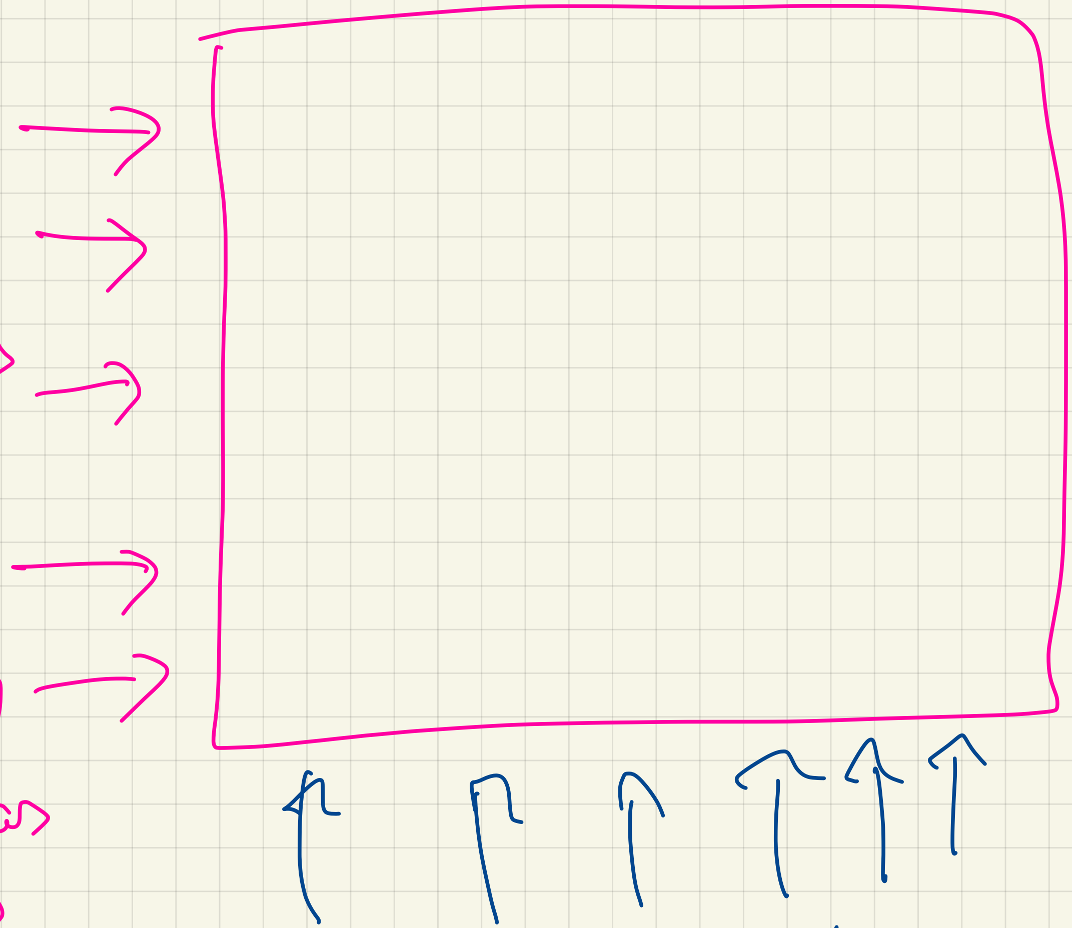
Col player q dist over cols

$$\boxed{A[p,q] = \sum_{i,j} p_i q_j \cdot A_{ij}}$$

$$\max_p \left[\min_q A[p,q] \right] = \min_q \left(\max_p A[p,q] \right)$$

n rows

Row player runs MWU with rows as experts



Col player also MWU with columns as experts