LINEAR PROGRAMMING

- DUALITY

- ZERO-SUM GAMES.

PROOFS OF UPPER BOUNDS ON OBJECTIVE

$$(2x_1 + x_2 \leq 100) \cdot 0 \quad \leftarrow \text{Flour}$$

$$(x_1 \leq 30) \cdot 5 \quad \leftarrow \text{Sugar}$$

$$(x_2 \leq 60) \cdot 4 \quad \leftarrow \text{Eggs}$$

$$5x_1 + 4x_2 \leq 5 \cdot 30 + 4 \cdot 60 = 390$$

$$\boxed{\begin{array}{l} x_1 \geq 0 \\ x_2 \geq 0 \end{array}}$$

Maximize $5x_1 + 4x_2$

$\text{OPT} \leq 390.$

$x_1 = \text{Bagel}$

$x_2 = \text{Donut}$

UPPER BOUNDS

$$
\begin{aligned}
(2x_1 + x_2 &\le 100)\cdot 3\\
(x_1 &\le 30)\cdot 0 \; +\\
(x_2 &\le 60)\cdot 1
\end{aligned}
$$

$x_1 \geq 0$

$x_2 \geq 0$

$$6x_1 + 4x_2 \le 3\cdot 100 + 60 = 360$$

$$\begin{matrix} \vee\!\!/ & & \vee\!\!/ \end{matrix}$$

Maximize $5x_1 + 4x_2$ $\quad$ OPT ≤ 360

$$
\begin{aligned}
(2x_1 + x_2 &\le 100)\cdot 5/2\\
(x_1 &\le 30)\cdot 0\\
(x_2 &\le 60)\cdot 3/2
\end{aligned}
$$

$x_1 \geq 0$

$x_2 \geq 0$

$$5x_1 + 4x_2 \le (100)\cdot 5/2 + 60\cdot(3/2)$$

$$= 340$$

Maximize $5x_1 + 4x_2$

$$\boxed{\text{OPT} \le 340}$$

LP (PRIMAL)

$$\begin{aligned}
(2x_1 + x_2 &\leq 100) \cdot y_1 \\
(x_1 &\leq 30) \cdot y_2 \\
(x_2 &\leq 60) \cdot y_3
\end{aligned}$$

$x_1 \geq 0$

$x_2 \geq 0$

$$\underset{\text{\rotatebox{90}{$\leq$}}}{(2y_1+y_2)} \cdot x_1 + \underset{\text{\rotatebox{90}{$\leq$}}}{(y_1+y_3)} \cdot x_2 \leq 100y_1 + 30y_2 + 60y_3$$

Maximize $5x_1 + 4x_2$

LP* (DUAL)

Minimise $100y_1 + 30y_2 + 60y_3$

$$\begin{aligned}
2y_1 + y_2 &\geq 5 \\
y_1 + y_3 &\geq 4
\end{aligned}$$

$y_1, y_2, y_3 \geq 0$

$$5x_1 + 4x_2 \leq (2y_1+y_2)x_1 + (y_1+y_3) \cdot x_2$$

$$\leq 100y_1 + 30y_2 + 60y_3$$

PRIMAL LP (Maximization)

$$\begin{bmatrix} & & \\ & A & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ x \\ \\ \end{bmatrix} \leq \begin{bmatrix} \\ b \\ \\ \end{bmatrix}$$

$$\text{Max} \begin{bmatrix} c^T \end{bmatrix} \begin{bmatrix} \\ x \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \\ x \\ \\ \end{bmatrix} \geq 0$$

DUAL LP (Minimization)

$$\begin{bmatrix} & & \\ & A^T & \\ & & \end{bmatrix} \cdot \begin{bmatrix} \\ y \\ \\ \end{bmatrix} \geq \begin{bmatrix} \\ c \\ \\ \end{bmatrix}$$

$$\text{Minimise} \begin{bmatrix} b^T \end{bmatrix} \begin{bmatrix} \\ y \\ \\ \end{bmatrix}$$

$$\begin{bmatrix} \\ y \\ \\ \end{bmatrix} \geq 0$$

DUAL LP

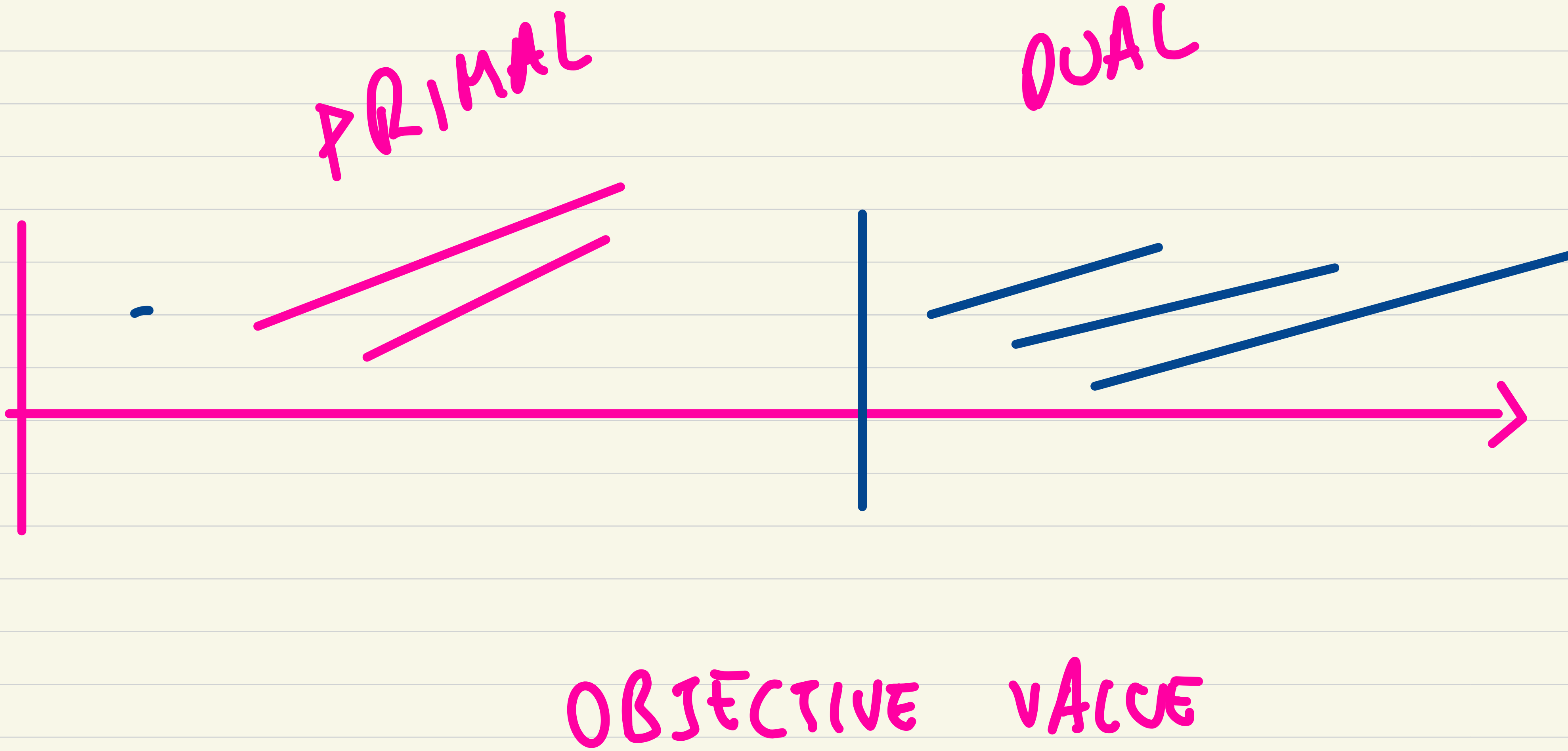
THM (WEAK DUALITY)

$\forall$ feasible solution x to PRIMAL Maximization LP $\leq$ "$\forall$" feasible solution y to "DUAL Minimization LP"

$$5x_1 + 4x_2 \leq 100y_1 + 30y_2 + 60y_3$$

$$\underbrace{5x_1 + 4x_2}_{\text{Value of solution "Primal"}} \leq (2y_1 + y_2)x_1 + (y_1 + y_3) \cdot x_2 \leq \underbrace{100y_1 + 30y_2 + 60y_3}_{\text{Value of solution "Dual LP"}}$$

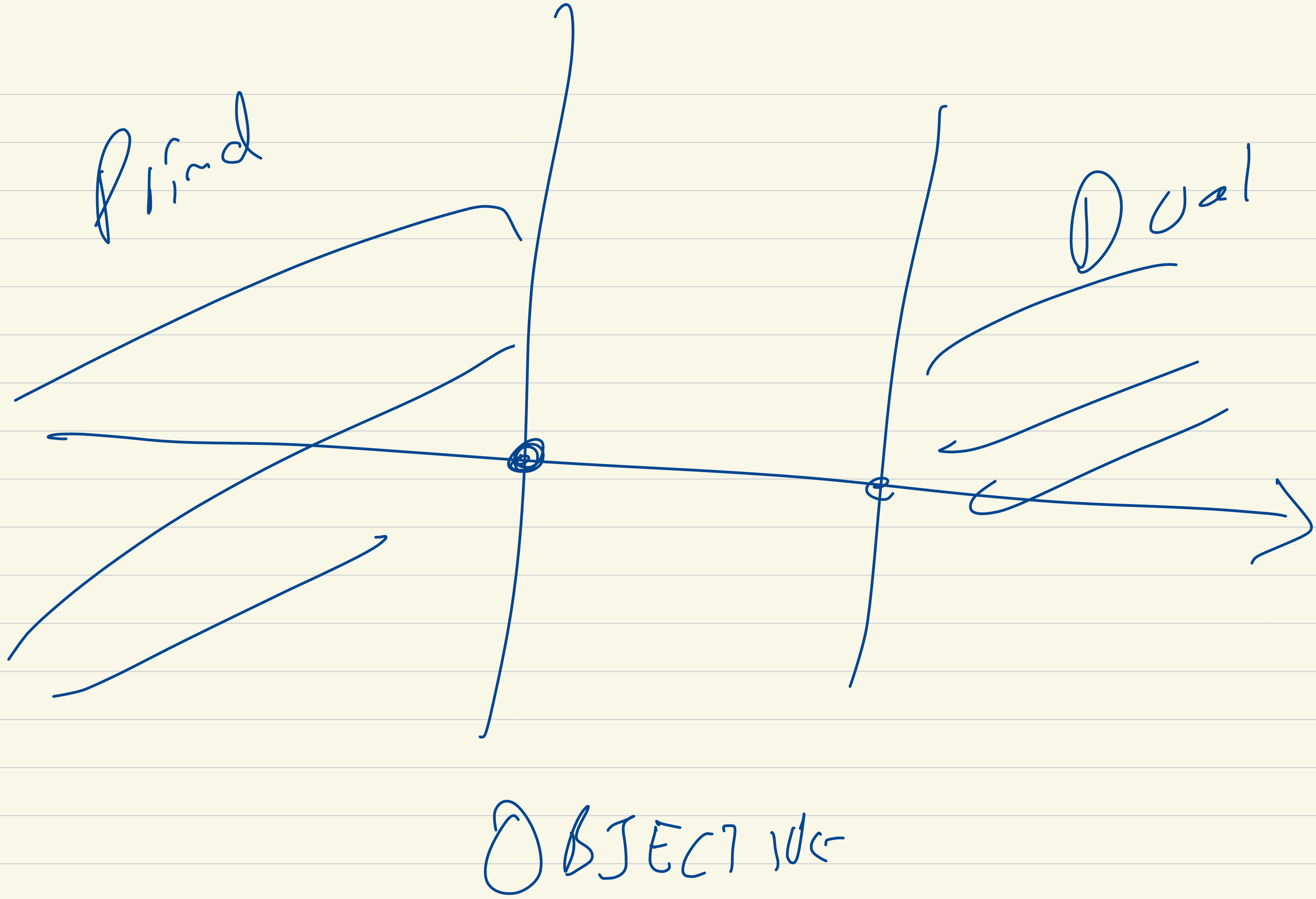
COROLLARY: PRIMAL OPT $\leq$ DUAL OPT



THM (STRONG CONVEXITY)

"If the PRIMAL OPT is bounded

PRIMAL OPT = DUAL OPT"


Primal
Dual
OBJECTIVE

ZERO-SUM GAMES

- A matrix $A[i,j]$
- Row player picks a row "r"
- Column player picks column "c"

Row player : $A[r,c]$

	ROCK	PAPER	SCISSOR
ROCK	0	-1	1
PAPER →	1	0	-1
SCISSOR	-1	1	0

ROW PLAYERS PAYOFF

TERMINOLOGY

PURE STRATEGY = a single row / single column

MIXED STRATEGY = a prob. distribution over "pure" strategies.

$$\left\{ Pr[Rock] = 1/2 \quad Pr[Paper] = 1/4 \quad Pr[Scissor] = 1/4 \right\}$$

GAME 1:

* Row announces mixed strategy p
* COLUMN plays second

	1	2
1	20	10
2	10	-30

GAME 2:

* COLUMN announces mixed strategy q
* Row plays second.

$$\max_{(\text{Row})}\left[\min_{(\text{Column})} \text{Expected Payoff}\right] \le \min_{(\text{Column})}\left[\max_{(\text{Row})}\left(\text{Expected Payoff}\right)\right]$$

(Row player is Better off in Game 2)

$$\begin{array}{ccc} \| & & \| \\ LP & \underset{\rightleftarrows}{(\text{Strong Duality})} & LP^* \end{array}$$

* Order of play <u>doesn't</u> change the Value.
* $\exists$ an optimal strategy for Row, for every column player strat.

GAME 1:

* Row announces mixed strategy P
* Column plays second

	1	2	
1	20	10	← p_1
2	10	-30	← p_2
	$20p_1+10p_2$	$10p_1-30p_2$	

$Pr[Row=1] = p_1$ $\qquad$ $Pr[Row=2] = p_2$ $\qquad$ $p_1+p_2=1$

$Pr[Column=1] = q_1$ $\qquad$ $Pr[Column=2] = q_2$ $\qquad$ $q_1+q_2=1$

VALUE OF A GAME = $\underset{\substack{\text{ROW, MIXED STRATEGIES} \\ p_1, p_2}}{\text{maximize}} \left[\underset{\substack{\text{COL MIXED STRATEGIES} \\ q_1, q_2}}{\text{minimize}} \left[20p_1 \cdot q_1 + 10p_1 \cdot q_2 + 10p_2q_1 - 30p_2q_2 \right] \right]$

expected Row Players Payoff

FACT:

$$\underset{\substack{\text{(ROW, MIXED STRATEGIES)} \\ p_1, p_2}}{\text{maximize}} \left[\underset{\substack{\text{(COL MIXED STRATEGIES)} \\ q_1, q_2}}{\text{minimize}} \left[20\, p_1 \cdot q_1 + 10\, p_1 \cdot q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$\Longrightarrow$$

$$\underset{\substack{\text{(ROW, MIXED STRATEGIES)} \\ p_1, p_2}}{\text{maximize}} \left[\underset{\substack{\text{(COL PURE STRATEGIES)} \\ q_1, q_2}}{\text{minimize}} \left[20\, p_1 \cdot q_1 + 10\, p_1 \cdot q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

PLAYER going SECOND can always be DETERMINISTIC

$$= \underset{\substack{\text{ROW's MIXED STRATEGIES}\\ p_1, p_2 \\ p_1+p_2=1}}{\text{maximize}} \left[\underset{\substack{\text{COL PURE STRATEGIES}\\ (q_1=1,\ q_2=0)\\ (q_1=0,\ q_2=1)}}{\text{minimize}} \left[20\,p_1 \cdot q_1 + 10\,p_1 \cdot q_2 + 10 p_2 q_1 - 30 p_2 q_2 \right] \right]$$

$$= \max_{p_1, p_2,\ p_1+p_2=1} \left[\underset{\substack{\text{over}\\ \text{the}\\ \text{two}}}{\min} \left\{ \begin{array}{l} 20p_1 + 10p_2, \\ 10p_1 - 30p_2 \end{array} \right\} \right]$$

$$\left\{ \max_{p_1, p_2,\ p_1+p_2=1} \left[\min \left[20p_1 + 10p_2\ ,\ 10p_1 - 30p_2 \right] \right] \right\}$$

$$\Updownarrow$$

LP

$$\boxed{\begin{aligned} &\text{Maximije} \quad z \\ &z \leq 20p_1 + 10p_2 \\ &z \leq 10p_1 - 30p_2 \\ &p_1 + p_2 = 1 \end{aligned}}$$

$$\parallel$$

VALUE OF GAME I

	1	2
1	20	10
2	10	-30

GAME 2:

* COLUMN announces mixed strategy q
* ROW plays second.

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* $\Pr[\text{COLUMN } 1] = q_1$ $\quad \Pr[\text{COLUMN } 2] = q_2$
* FOR ROW PLAYER: Expected PAYOFF for ROW 1 $= 20q_1 + 10q_2$

  Expected PAYOFF for ROW 2 $= 10q_1 - 30q_2$

$$\text{VALUE}(\text{GAME } 2) = \min_{\substack{\text{COLUMN MIXED STRATEGIES} \\ q_1+q_2=1}} \left[\max_{\text{ROW MIXED/STRATEGIES}} \left[20p_1 \cdot q_1 + 10p_1 \cdot q_2 + 10p_2 q_1 - 30p_2 q_2 \right] \right]$$

$$= \min_{q_1+q_2=1} \left[\max\{20q_1 + 10q_2,\ 10q_1 - 30q_2\} \right]$$

$$\min_{q_1, q_2,\, q_1+q_2=1} \left[\max\{20q_1+10q_2,\ 10q_1-30q_2\} \right]$$

$$\Updownarrow$$

LP*

$$\boxed{\begin{aligned} \text{Min} \quad & Z \\ & 20q_1 + 10q_2 \leq Z \\ & 10q_1 - 30q_2 \leq Z \\ & q_1 + q_2 = 1 \\ & q_1, q_2 \geq 0 \end{aligned}}$$