Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Basics

Flow. The *capacity* indicates how much flow can be allowed on an edge. Given a directed graph with edge capacity $c(u, v)$ and s, t, a flow is a mapping $f : E \to \mathbb{R}^+$ that satisfies

- Capacity constraint: $f(u, v) \leq c(u, v)$, the flow on an edge cannot exceed its capacity.
- Conservation of flows: $f^{\text{in}}(v) = f^{\text{out}}(v)$, flow in equals flow out for any $v \notin \{s, t\}$

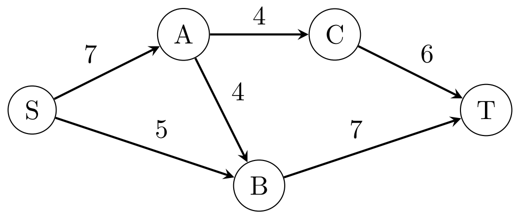
Here, we define $f^{\text{in}}(v) = \sum_{u:(u,v)\in E} f(u, v)$ and $f^{\text{out}}(v) = \sum_{u:(v,u)\in E} f(u, v)$. We also define $f(v, u) = -f(u, v)$, and this is called *skew-symmetry*.

Residual Graph. Given a flow network (G, s, t, c) and a flow f, the *residual capacity* (w.r.t. flow f) is denoted by $c_f(u, v) = c_{uv} - f_{uv}$. And the *residual network* $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$.

Ford-Fulkerson. Keep pushing along s-t paths in the residual graph. Runs in time $O(mF)$.

2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

3 Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph G and the additional variant constraints, show how to construct a directed graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v)\in E} f_{uv} \leq c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

4 Provably Optimal

Consider the following linear program:

$$\begin{aligned} \max\ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

For the linear program above,

(a) First compute the dual of the above linear program

(b) show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual. (*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

5 Taking a Dual

Consider the following linear program:

$$\begin{aligned} \max\ & 4x_1 + 7x_2 \\ & x_1 + 2x_2 \leq 10 \\ & 3x_1 + x_2 \leq 14 \\ & 2x_1 + 3x_2 \leq 11 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Construct the dual of the above linear program.