Note: Your TA probably will not cover all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. They are deliberately made long so they can serve as a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Basics

Flow. The *capacity* indicates how much flow can be allowed on an edge. Given a directed graph with edge capacity $c(u, v)$ and s, t, a flow is a mapping $f : E \to \mathbb{R}^+$ that satisfies

- Capacity constraint: $f(u, v) \leq c(u, v)$, the flow on an edge cannot exceed its capacity.
- Conservation of flows: $f^{\text{in}}(v) = f^{\text{out}}(v)$, flow in equals flow out for any $v \notin \{s, t\}$

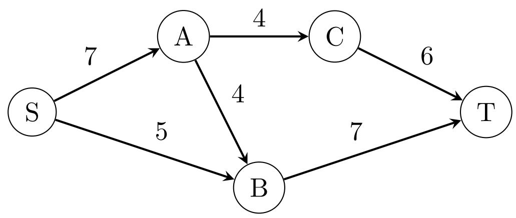
Here, we define $f^{\text{in}}(v) = \sum_{u:(u,v)\in E} f(u, v)$ and $f^{\text{out}}(v) = \sum_{u:(v,u)\in E} f(u, v)$. We also define $f(v, u) = -f(u, v)$, and this is called *skew-symmetry*.

Residual Graph. Given a flow network (G, s, t, c) and a flow f, the *residual capacity* (w.r.t. flow f) is denoted by $c_f(u, v) = c_{uv} - f_{uv}$. And the *residual network* $G_f = (V, E_f)$ where $E_f = \{(u, v) : c_f(u, v) > 0\}$.

Ford-Fulkerson. Keep pushing along s-t paths in the residual graph. Runs in time $O(mF)$.

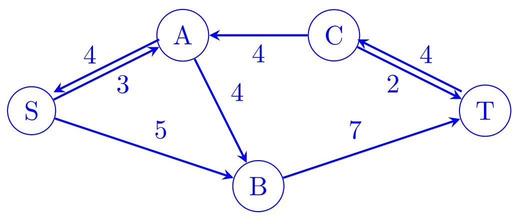
2 Residual in graphs

Consider the following graph with edge capacities as shown:



(a) Consider pushing 4 units of flow through $S \to A \to C \to T$. Draw the residual graph after this push.

Solution:

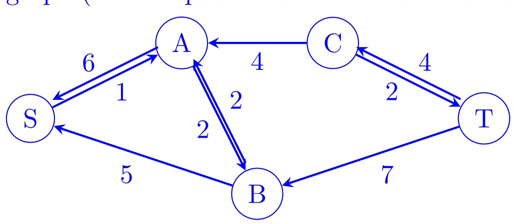


(b) Compute a maximum flow of the above graph. Find a minimum cut. Draw the residual graph of the maximum flow.

Solution: A maximum flow of value 11 results from pushing:

- 4 units of flow through $S \to A \to C \to T$;
- 5 units of flow through $S \to B \to T$; and
- 2 units of flow through $S \to A \to B \to T$.

(There are other maximum flows of the same value, can you find them?) The resulting residual graph (with respect to the maximum flow above) is:



A minimum cut of value 11 is between $\{S, A, B\}$ and $\{C, T\}$ (with cross edges $A \to C$ and $B \to T$).

3 Reductions Among Flows

Show how to reduce the following variants of Max-Flow to the regular Max-Flow problem, i.e. do the following steps for each variant: Given a directed graph G and the additional variant constraints, show how to construct a directed graph G' such that

(1) If F is a flow in G satisfying the additional constraints, there is a flow F' in G' of the same size,

(2) If F' is a flow in G', then there is a flow F in G satisfying the additional constraints with the same size.

Prove that properties (1) and (2) hold for your graph G'.

(a) **Max-Flow with Vertex Capacities:** In addition to edge capacities, every vertex $v \in G$ has a capacity c_v, and the flow must satisfy $\forall v : \sum_{u:(u,v)\in E} f_{uv} \leq c_v$.

(b) **Max-Flow with Multiple Sources:** There are multiple source nodes $s_1, \ldots, s_k$, and the goal is to maximize the total flow coming out of all of these sources.

Solution:

(a) Split every vertex v into two vertices, v_{in} and v_{out}. For each edge (u, v) with capacity c_{uv} in the original graph, create an edge (u_{out}, v_{in}) with capacity c_{uv}. Finally, if v has capacity c_v, then create an edge (v_{in}, v_{out}) with capacity c_v. If F' is a flow in this graph, then setting $F(u, v) = F'(u_{out}, v_{in})$ gives a flow in the original graph. Moreover, since the only outgoing edge from v_{in} is (v_{in}, v_{out}), and incoming flow must be equal to outgoing flow, there can be at most c_v flow passing through v. Likewise, if F is a flow in the original graph, setting $F'(u_{out}, v_{in}) = F(u, v)$, and $F'(v_{in}, v_{out}) = \sum_u F(u, v)$ gives a flow in G'. One can easily see that these flows have the same size.

(b) Create one "supersource" S with edges (S, s_i) for each s_i, and set the capacity of these edges to be infinite. Then if F is a flow in G, set $F'(S, s_i) = \sum_u F(s_i, u)$. Conversely, if F' is a flow in G', just set $F(u, v) = F'(u, v)$ for $u \neq S$, and just forget about the edges from S. One can easily see that these flows have the same size.

4 Provably Optimal

Consider the following linear program:

$$\begin{aligned} \max\ & x_1 - 2x_3 \\ & x_1 - x_2 \leq 1 \\ & 2x_2 - x_3 \leq 1 \\ & x_1, x_2, x_3 \geq 0 \end{aligned}$$

For the linear program above,

(a) First compute the dual of the above linear program

(b) show that the solution $(x_1, x_2, x_3) = (3/2, 1/2, 0)$ is optimal **using its dual**. You do not have to solve for the optimum of the dual. (*Hint:* Recall that any feasible solution of the dual is an upper bound on any feasible solution of the primal)

Solution: The dual of the given LP is:

$$\begin{aligned} \min\ & y_1 + y_2 \\ & y_1 \geq 1 \\ & -y_1 + 2y_2 \geq 0 \\ & -y_2 \geq -2 \\ & y_1, y_2 \geq 0 \end{aligned} \qquad \equiv \qquad \begin{aligned} \min\ & y_1 + y_2 \\ & y_1 \geq 1 \\ & y_2 \geq \frac{y_1}{2} \\ & y_2 \leq 2 \\ & y_1, y_2 \geq 0 \end{aligned}$$

The objective value at the claimed optimum is $3/2$. By the duality theorem, this would be optimum if and only if there is a feasible solution to the dual LP with the same objective value. Greedily trying to make y_1, y_2 as small as possible results in finding that$y_1 = 1, y_2 = 1/2$ is a feasible dual solution, with the objective value $3/2$. Thus, the claimed primal optimal is indeed an optimal solution.

5 Taking a Dual

Consider the following linear program:

$$\begin{aligned} &\max\ 4x_1 + 7x_2 \\ x_1 + 2x_2 &\leq 10 \\ 3x_1 + x_2 &\leq 14 \\ 2x_1 + 3x_2 &\leq 11 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Construct the dual of the above linear program.

Solution: If we scale the first constraint by $y_1 \geq 0$, the second by $y_2 \geq 0$, the third by $y_3 \geq 0$, and we add them up, we get an upperbound of $(y_1+3y_2+2y_3)x_1+(2y_1+y_2+3y_3)x_2 \leq (10y_1+14y_2+11y_3)$. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned} &\min\ 10y_1 + 14y_2 + 11y_3 \\ y_1 + 3y_2 + 2y_3 &\geq 4 \\ 2y_1 + y_2 + 3y_3 &\geq 7 \\ y_1, y_2, y_3 &\geq 0 \end{aligned}$$