CS 170 HW 8

Due **2020-10-26, at 10:00 pm**

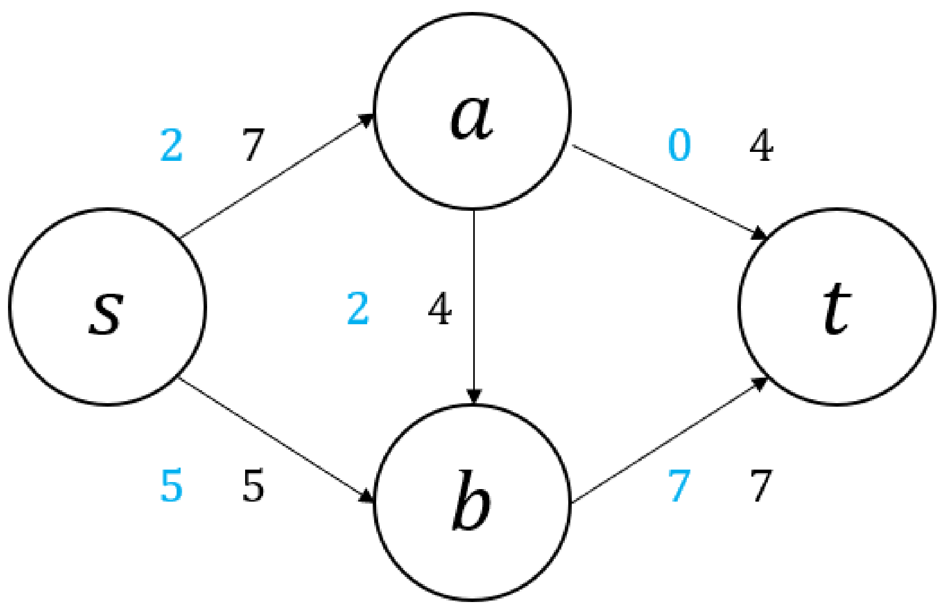
1 Study Group

List the names and SIDs of the members in your study group. If you have no collaborators, you must explicitly write none.

In addition, we would like to share correct student solutions that are well-written with the class after each homework. Are you okay with your correct solutions being used for this purpose? Answer "Yes", "Yes but anonymously", or "No"

2 Practice With Residual Graphs

(a) Consider the following network and flow on this network. An edge is labelled with its flow value (in blue) and capacity (in black). e.g. for the edge (s, a), we are currently pushing 2 units of flow on it, and it has capacity 7.



Draw the residual graph for this flow.

(b) We are given a network $G = (V, E)$ whose edges have integer capacities c_e, and a maximum flow f from node s to node t. Explicitly, f is given to us in the representation of integer flows along every edge e, (f_e).

However, we find out that one of the capacity values of G was wrong: for edge (u, v), we used c_{uv} whereas it really should have been $c_{uv} - 1$. This is unfortunate because the flow f uses that particular edge at full capacity: $f_{uv} = c_{uv}$. We could run Ford Fulkerson from scratch, but there's a faster way.

Describe an algorithm to fix the max-flow for this network in $O(|V| + |E|)$ time. Give a three-part solution.

3 Global Min-Cut via Ford-Fulkerson

Given a connected undirected unweighted graph G, recall that a cut is any partition of V into two non-empty sets A, B, and the size of the cut is the number of edges with one endpoint in A and one in B. The global min-cut is the cut with the smallest size.

Give an algorithm based on the Ford-Fulkerson algorithm that given G and c, either outputs the global min-cut if it has size at most c, or correctly outputs that the global min-cut has size greater than c. The algorithm should run in time $O(|V||E|c)$. Give a three-part solution.

(Recall that the Ford-Fulkerson algorithm repeatedly finds an s-t path in the residual graph and pushes as much flow as possible on this path)

4 Meal Replacement

We are trying to eat cheaply but still meet our minimum dietary needs. We want to consume at least 500 calories of protein per day, 100 calories of carbs per day, and 400 calories of fat per day. We have three options for food we're considering buying: meat, bread, and protein shakes.

- We can consume meat, which costs 5 dollars per pound, and gives 500 calories of protein and 500 calories of fat per pound.
- We can consume bread, which costs 2 dollars per pound, and gives 50 calories of protein, 300 calories of carbs, and 25 calories of fat per pound.
- We can consume protein shakes, which cost 4 dollars per pound, and gives 300 calories of protein, 100 calories of carbs, and 200 calories of fat per pound.

Our goal is to find a combination of these options that meets our daily dietary needs while being as cheap as possible.

(a) Formulate this problem as a linear program.

(b) Take the dual of your LP from part (a).

(c) Suppose now there is a pharmacist trying to assign a price to three pills, with the hopes of getting us to buy these pills instead of food. Each pill provides exactly one of protein, carbs, and fiber.

Interpret the dual LP variables, objective, and constraints as an optimization problem from the pharmacist's perspective.

5 Vertex Cover Dual

This is a solo question.

In this problem, we consider the unweighted vertex cover problem. In this problem, we are given a graph and want to find the smallest set of vertices S such that every edge has at least one endpoint in S.

Recall the LP for this problem:

$$\min \sum_v x_v \text{s. t. } \forall (u,v) \in E : x_u + x_v \geq 1, \forall v \in V : x_v \geq 0$$

In an integral solution, $x_v = 1$ if we include v in our vertex cover, and $x_v = 0$ if we don't include v.

(i) Write the dual of the vertex cover LP. Your dual LP should have a variable y_e for every edge.

(ii) Consider the integer version of the dual you wrote, i.e. we enforce $y_e \in \{0, 1\}$. Similarly to vertex cover, we can interpret $y_e = 1$ as indicating that we include e in our solution and $y_e = 0$ if we don't include e.

Using this interpretation, what does the objective say? What do the constraints say? What problem is this? (You don't have to be formal.)

(iii) True or False: If we have an integer primal solution with cost C and a fractional dual solution with cost at least $C/2$, the size of the vertex cover corresponding to the primal solution is at most twice the size of the smallest vertex cover. Briefly justify your answer.

6 Domination

This is a solo question.

In this problem, we explore a concept called *dominated strategies*. Consider a zero-sum game with the following payoff matrix for the row player:

		Column: A	B	C
		A	B	C
Row:	D	1	2	-3
	E	3	2	-2
	F	-1	-2	2

(a) If the row player plays optimally, can you find the probability that they pick D without directly solving for the optimal strategy? Justify your answer.

(Hint: How do the payoffs for the row player picking D compare to their payoffs for picking E?)

(b) Given the answer to part a, if the both players play optimally, what is the probability that the column player picks A?

(c) Given the answers to part a and b, what are both players' optimal strategies?

Note: All parts of this problem can be solved without using an LP solver or solving a system of linear equations.

7 Zero-Sum Battle

This is a solo question.

Two Pokemon trainers are about to engage in battle! Each trainer has 3 Pokemon, each of a single, unique type. They each must choose which Pokemon to send out first. Of course each trainer's advantage in battle depends not only on their own Pokemon, but on which Pokemon their opponent sends out.

The table below indicates the competitive advantage (payoff) Trainer A would gain (and Trainer B would lose). For example, if Trainer B chooses the fire Pokemon and Trainer A chooses the rock Pokemon, Trainer A would have payoff 2.

		Trainer B:		
		ice	water	fire
	dragon	-10	3	3
Trainer A:	steel	4	-1	-3
	rock	6	-9	2

Feel free to use an online LP solver to solve your LPs in this problem.
Here is an example of an online solver that you can use: `https://online-optimizer.appspot.com/`.

1. Write an LP to find the optimal strategy for Trainer A. What is the optimal strategy and expected payoff?
2. Now do the same for Trainer B. What is the optimal strategy and expected payoff? How does the expected payoff compare to the answer you get in part (a)?