Note: Your TA may not get to all the problems. This is totally fine, the discussion worksheets are not designed to be finished in an hour. The discussion worksheet is also a resource you can use to practice, reinforce, and build upon concepts discussed in lecture, readings, and the homework.

1 Taking a Dual

Consider the following linear program:

$$\begin{aligned} &\max\ 4x_1 + 7x_2 \\ x_1 + 2x_2 &\le 10 \\ 3x_1 + x_2 &\le 14 \\ 2x_1 + 3x_2 &\le 11 \\ x_1, x_2 &\ge 0 \end{aligned}$$

Construct the dual of the above linear program.

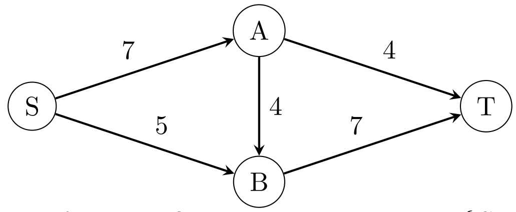
Solution: If we scale the first constraint by $y_1 \ge 0$, the second by $y_2 \ge 0$, the third by $y_3 \ge 0$, and we add them up, we get an upperbound of $(y_1+3y_2+2y_3)x_1+(2y_1+y_2+3y_3)x_2 \le (10y_1+14y_2+11y_3)$. We need y_1, y_2, y_3 to be non-negative, otherwise the signs in the inequalities flip. Minimizing for a bound for $4x_1 + 7x_2$, we get the tightest possible upperbound by

$$\begin{aligned} &\min\ 10y_1 + 14y_2 + 11y_3 \\ y_1 + 3y_2 + 2y_3 &\ge 4 \\ 2y_1 + y_2 + 3y_3 &\ge 7 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

2 Max Flow, Min Cut, and Duality

In this exercise, we will demonstrate that LP duality can be used to show the max-flow min-cut theorem.

Consider this instance of max flow:



Let f_1 be the flow pushed on the path $\{S, A, T\}$, f_2 be the flow pushed on the path $\{S, A, B, T\}$, and f_3 be the flow pushed on the path $\{S, B, T\}$. The following is an LP for max flow in terms of the variables f_1, f_2, f_3:

$$
\begin{aligned}
\max \quad & f_1 + f_2 + f_3 & \\
& f_1 + f_2 \leq 7 & \text{(Constraint for } (S, A))\\
& f_3 \leq 5 & \text{(Constraint for } (S, B))\\
& f_1 \leq 4 & \text{(Constraint for } (A, T))\\
& f_2 \leq 4 & \text{(Constraint for } (A, B))\\
& f_2 + f_3 \leq 7 & \text{(Constraint for } (B, T))\\
& f_1, f_2, f_3 \geq 0 &
\end{aligned}
$$

The objective is to maximize the flow being pushed, with the constraint that for every edge, we can't push more flow through that edge than its capacity allows.

(a) Find the dual of this linear program, where the variables in the dual are x_e for every edge e in the graph.

(b) Consider any cut in the graph. Show that setting $x_e = 1$ for every edge crossing this cut and $x_e = 0$ for every edge not crossing this cut gives a feasible solution to the dual program.

(c) Based on your answer to the previous part, what problem is being modelled by the dual program? By LP duality, what can you argue about this problem and the max flow problem?

Solution:

(a) The dual is:

$$
\begin{aligned}
\min \quad & 7x_{SA} + 5x_{SB} + 4x_{AT} + 4x_{AB} + 7x_{BT} \\
x_{SA} + x_{AT} \geq 1 & \quad \text{- Constraint for } f_1 \\
x_{SA} + x_{AB} + x_{BT} \geq 1 & \quad \text{- Constraint for } f_2 \\
x_{SB} + x_{BT} \geq 1 & \quad \text{- Constraint for } f_3 \\
x_e \geq 0 & \quad \forall e \in E
\end{aligned}
$$

(b) Notice that each constraint contains all variables x_e for every edge e in the corresponding path. For any $s-t$ cut, every $s-t$ path contains an edge crossing this cut. So for any cut, the suggested solution will set at least one x_e to 1 on each path, giving that each constraint is satisfied.

(c) The dual LP is an LP for the min-cut problem. By the previous answer, we know the constraints describe solutions corresponding to cuts. The objective then just says to find the cut of the smallest size. By LP duality, the dual and primal optima are equal, i.e. the max flow and min cut values are equal.

3 Dual of Maximum Independent Set

You are given a connected undirected graph $G = (V, E)$ where $|V| > 2$. Recall that a set of vertices $S \subseteq V$ is an *independent set* if there do not exist $u, v \in S$ such that there is an edge between u and v. In additon, an *edge cover* is a set of edges $C \subseteq E$ such that for each vertex v, there is some edge in C that it is incident to (so the edges in C 'cover' all the vertices).

(a) In the *maximum independent set* problem, you want to find an independent set of maximum size. Write the integer linear program (ILP) for the relaxed version of maximum independent set (that is, you may have the constraint $x \in \{0, 1\}$).

Solution: The ILP for maximum independent set is as follows:

$$\begin{aligned} \max \quad & \sum_{v \in V} x_v \\ \text{subject to: for all } (u,v) \in E\text{:} \quad & x_u + x_v \leq 1 \\ \text{for all } v \in V\text{:} \quad & x_v \in \{0,1\} \end{aligned}$$

(b) Take your ILP, and replace the constraints of the form $x \in \{0, 1\}$ with $x \geq 0$ to get a linear program (LP). Then find the dual LP of this LP. What problem does the dual represent?

Solution: The dual is as follows:

$$\begin{aligned} \min \quad & \sum_{(u,v) \in E} y_{uv} \\ \text{subject to: for all } v \in E\text{:} \quad & \sum_{(v,u) \in E} y_{vu} \geq 1 \\ \text{for all } (u,v) \in V\text{:} \quad & y_{uv} \geq 0 \end{aligned}$$

The dual is (the fractional version of) the minimum edge cover problem.

(c) True or false: For any connected graph, the optimum value for the (non-integer) primal-dual pair you constructed in part (b) are always equal. If true, prove. If false, give a counterexample.

Solution: True. This just follows from strong duality for LPs, if we can show the primal optimal is bounded. The primal optimal is bounded because every vertex v has an edge leading out of it, so $x_v \leq 1$, so the sum is at most $|V|$.

(d) Take the ILP from part (a), and consider the ILP formed from the dual you found in part (b) by forcing all variables to be integers. True or false: The optimum values of these two ILPs are always equal.

Solution: False. The tightened version of the ILPs are maximum independent set and minimum edge cover.

Consider K_3, the three-vertex graph with an edge between every pair. The maximum independent set has size 1, while the smallest edge cover has size 2. So this is a countexample to the statement.

4 A Path and Edge Game

Edith and Paris are playing the following game: We have a unweighted undirected graph G, and a source and sink s, t. Edith picks an edge in G, and Paris picks a path from s to t in G. Edith wins if she picks an edge in Paris's path, and Paris wins if this does not happen.

Both players are allowed to use randomized strategies, i.e. they choose a distribution of edges/-paths respectively, and pick their edge/path from this distribution.

(a) Suppose Edith announces her distribution over edges to Paris, and then Paris gets to pick a distribution over paths based on Edith's distribution. They then each sample from their distributions and decide who wins based on the results.

Briefly argue that if Paris knows Edith's distribution, there is always an optimal strategy for Paris where he deterministically picks a single path.

(b) Consider two variants of the game:

- In Variant 1, Edith picks a distribution over edges and announces it to Paris. Paris then picks a single path.
- In Variant 2, Paris picks a distribution over paths and announces it to Edith. Edith then picks a single edge.

Fill in the blank: Regardless of how Paris plays, if Edith plays optimally, her chances of winning in Variant ___ are at least her chances of winning in the other Variant. (You don't need to formally justify your answer)

(c) Show that in Variant 1, Edith has a strategy that wins with probability at least $1/C$ regardless of Paris's strategy, where C is the size of the minimum s-t cut in G.

(d) Show that in Variant 2, Paris has a strategy such that Edith wins with probability at most $1/F$ regardless of Edith's strategy, where F is the size of the max flow s-t flow in G. (Hint: Any max flow can be decomposed into a set of paths p, where we push f_p flow on the path p, such that $f(e) = \sum_{p:e\in p} f_p$ and $\sum_p f_p = F$.)

(e) Based on your answers to the previous parts, if both Edith and Paris play optimally, is Edith's chance of winning in Variant 1 greater than, equal or, or less than her chance of winning in Variant 2?

Solution:

(a) Let W_p be the probability Paris wins if he picks the path p given Edith's strategy. For any randomized strategy Paris uses, $\mathbb{E}_p[W_p] \le \max_p W_p$, i.e. his chances of winning using the randomized strategy are at most the chance he wins if he just picks the best single path.

(b) Edith's chances of winning in Variant 2 are at least her chances of winning in Variant 1. She has an advantage in variant 2 in that she gets to choose an edge in response to Paris's strategy, rather than Paris getting to respond to hers. Notice that in part (a), we showed her only playing determinstic strategies in variant 2 isn't actually a disadvantage.

(c) Edith picks an edge uniformly at random from the minimum s-t cut. Paris must choose a path containing at least one of these edges, which means regardless of what path Paris chooses, Edith chooses an edge in that path with probability at least $1/C$.

(d) Paris finds a max-flow F, decomposes it into path flows, and then picks each path with probability f_p/F. Edith's best response is to pick the edge such that the chance this edge is in the path, $\sum_{p:e\in p} f_p/F$, is maximized. But the capacity constraints on flow guarantee that $\sum_{p:e\in p} f_p \le 1$, so Edith's chance of winning is at most $1/F$.

(e) Equal. (c), (d) show Edith's chances of winning in both variants are equal by max-flow min-cut theorem.

Note: As you will learn in lecture, the problem of solving for Paris and Edith's optimal strategies can be expressed as a primal-dual linear program pair. In part (b), you effectively argued for weak duality of these problems, and in part (e) you argue for strong duality.