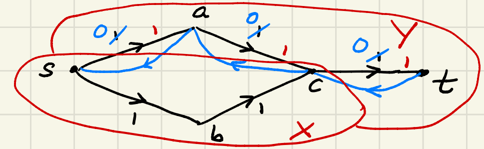
1. Max Flow – Min Cut

2. LP Duality

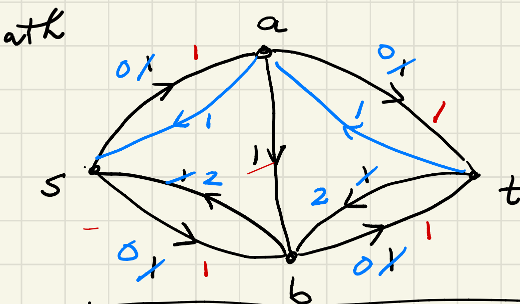
3. Two person zero sum games

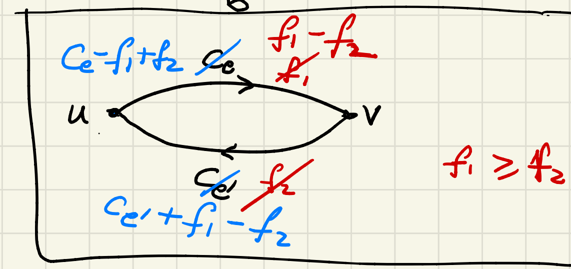



Find a path from s to t

Push as much flow as possible through path

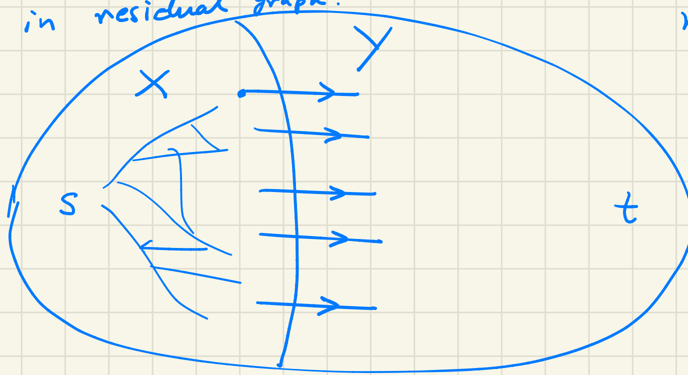
Update residual graph




Halts with max flow + certificate of optimality
$\|$
min cut.

X = vertices reachable from s in residual graph.




All edges $(u,v)$ $u \in X$ $v \in Y$

$$f_{uv} = c_{uv}$$

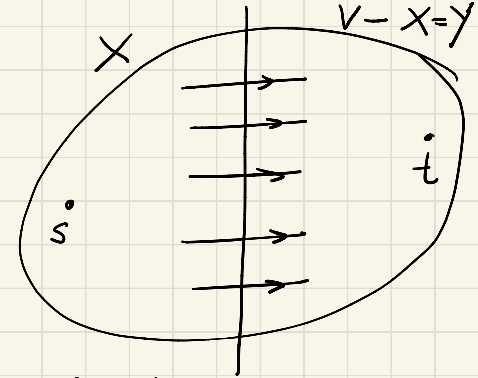
$$\sum_{\substack{uv \\ u \in X\ v \in Y}} f_{uv} = \text{value}(f) = \sum_{u,v} c_{u,v}$$

$$= \text{cap}(X, Y).$$

$\therefore$ $f =$ maxflow & $(X, Y) =$ min cut

Graph $G = (V, E)$, $s, t$
with capacities $c_e$

$$\text{value}(f) = \sum_{(s,u) \in E} f_{s,u} = \sum_{(v,t) \in E} f(v,t)$$




$$\text{value}(f) \le \text{cap}(X, Y)$$

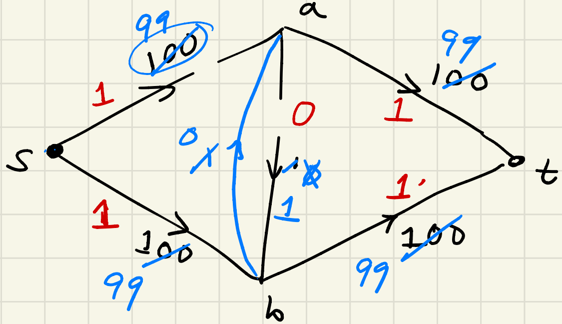
$$\text{cap}(X,Y) = \sum_{\substack{(u,v) \in E \\ u \in X \\ v \in Y}} c_{uv}$$

value$(f)$ = flow leaving $s$

= flow crossing from $X$ to $Y$

$\therefore$ max flow $\le$ min-cut

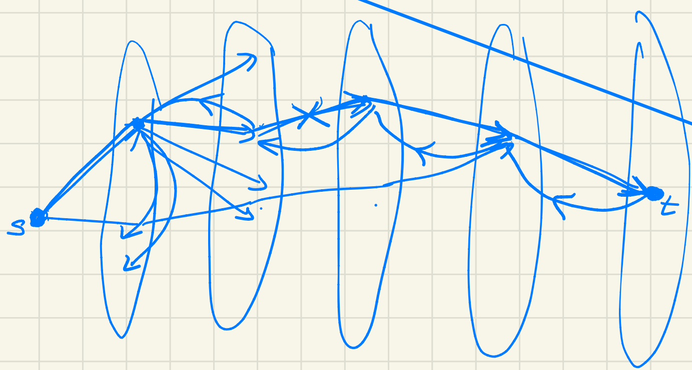
Max flow algorithm: max flow = min cut.




200 steps

Network Flow: shortest paths

$O(nm) \Rightarrow O(nm^2)$




# shortest paths $= O(nm)$. } $O(m)$ paths of length $d$.

Time to find one shortest path

$O(m)$

$\therefore$ total time $= O(nm^2)$.

# LP duality:

$\max \quad P + 2B.$

| | |
|---|---|
| $x$ | $P \le 400$ |
| $y$ | $B \le 300$ |
| $z$ | $2P + 3B \le 1200$ |

$P, B \ge 0$

Solution : $P = 150$
$B = 300$

profit $= 750$

$2P + 4B \le 1500$

$P + 2B \le 750$

$P + 2B \le$ ____

$\underline{\underline{P+2B}} \le 2P + 2B \le 1000$

$xP \le 400x \qquad x \ge 0$

$yB \le 300y. \qquad y \ge 0$

$z(2P + 3B) \le 1200z. \qquad z \ge 0$

$(x + 2z)P + (y + 3z)B \le 400x + 300y + 1200z.$

$P + 2B \le$ minimize.

$$\boxed{\begin{aligned} & x + 2z \ge 1 \\ & y + 3z \ge 2. \qquad x, y, z \ge 0 \\ & \min\ 400x + 300y + 1200z \end{aligned}}$$

Dual LP.

$x = 0 \quad y = \frac{1}{2}, \quad z = \frac{1}{2}.$

$\frac{1}{2} \cdot 300 + \frac{1}{2} 1200 = 750$

solution to LP $=$ ~~max~~ Any solution to dual.

max LP min dual

Primal

Dual.

$$\max \quad c_1 x_1 + \cdots + c_n x_n$$

$$a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1$$
$$a_{21} x_1 + \cdots + a_{2n} x_n \leq b_2$$
$$\vdots$$
$$a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m$$

$$x_1, x_2, \ldots, x_n \geq 0$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad A = \begin{pmatrix} a_{11} & & a_{1n} \\ & & \\ a_{m1} & & a_{mn} \end{pmatrix}$$

$$\vec{c} = \begin{pmatrix} c_1 \\ \\ c_n \end{pmatrix}$$

$$\vec{c}^T = (c_1 \; \cdots \; c_n) \begin{pmatrix} x \\ \\ x_n \end{pmatrix}$$

$$\boxed{\begin{array}{l} \max \; \vec{c}^T \vec{x} \\ \quad A\vec{x} \leq \vec{b} \\ \quad \vec{x} \geq 0 \end{array}}$$

Primal

$$\max \quad c_1 x_1 + \cdots + c_n x_n$$

$$\begin{array}{ll}
y_1 : & a_{11} x_1 + \cdots + a_{1n} x_n \le b_1 \\
y_2 : & a_{21} x_1 + \cdots + a_{2n} x_n \le b_2 \\
 & \vdots \\
y_m : & a_{m1} x_1 + \cdots + a_{mn} x_n \le b_m
\end{array}$$

$$x_1, x_2, \ldots, x_n \ge 0$$

Dual.

$$\min \quad b_1 y_1 + \cdots + b_m y_m$$

$$\begin{array}{ll}
x_1 : & y_1 a_{11} + \cdots + y_m a_{m1} \ge c_1 \\
x_2 : & y_1 a_{12} + \cdots + y_m a_{m2} \ge c_2 \\
 & \vdots \\
x_n : & y_1 a_{1n} + \cdots + y_m a_{mn} \ge c_n
\end{array}$$

$$y_1, \ldots, y_m \ge 0$$

$$\max \ \vec{c}^T \vec{x}$$

$$A \vec{x} \le \vec{b}$$

$$\vec{x} \ge 0$$

$$\min \ \vec{b} \cdot \vec{y}$$

$$\vec{y}^T A \ge \vec{c}^T$$

$$\vec{y} \ge 0$$

$$(y_1 \ y_2 \ \cdots \ y_m) \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$$