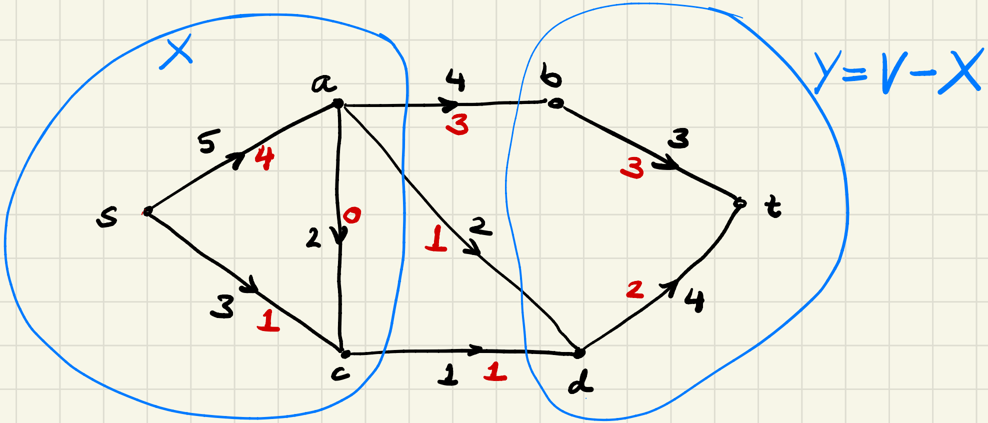
Max Flow & Min Cut



max flow problem: maximum flow through network subject to capacity constraints.

flow conservation: flow entering vertex = flow leaving vertex.

max flow ≤ min cut

=

min cut

$X = \{s, a, c\}$

$Y = \{b, d, t\}$

s-t cut: $s \in X$, $t \in Y$

$$cap(X,Y) = \sum_{\substack{x \in X \\ y \in Y}} c_{x,y}$$

$$cap(X,Y) = 4+2+1 = 7$$

* Max flow and min cut can be written as linear programs. } reduction

* Algorithm for max flow $\longleftrightarrow$ simplex algorithm

* Relationship between max flow and min cut

LP duality

Writing max flow as a linear program:

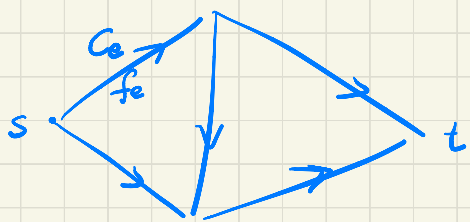
variable $f_{u,v}$ for every edge $(u,v) = e$ $\qquad$ $f_{u,v} = f_e$

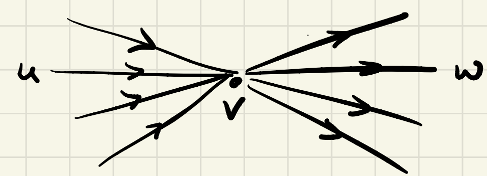
Objective function: maximize total flow = flow leaving s = flow entering t

$$\max \sum_{(s,u)\in E} f_{s,u}$$

Capacity Constraints $\quad \forall e \in E \quad 0 \le f_e \le c_e$

Flow Conservation $\quad \forall v \in V - \{s,t\}: \quad \sum_{(u,v)\in E} f_{uv} = \sum_{(v,w)\in E} f_{vw}$



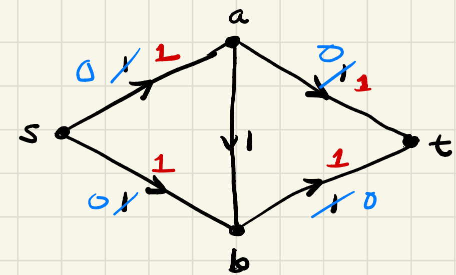


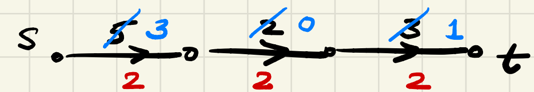
Max Flow Algorithm:

* Route a flow path:
  Find a path from s to t
  route max flow through this path

* Update capacities by updating residual graph.

repeat

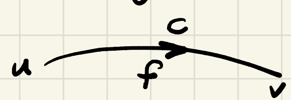




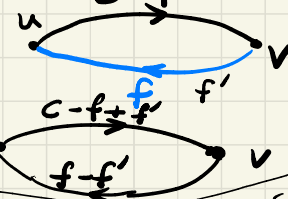
Ex s — a — t 1
s — b — t 1
max flow = 2.

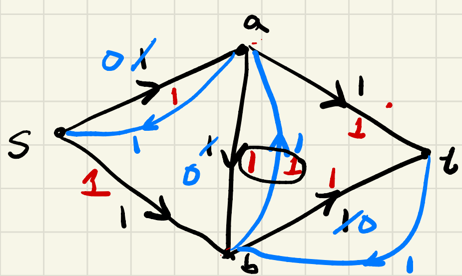
Residual graph:

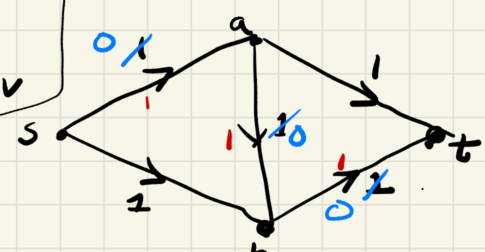
edge



update capacity





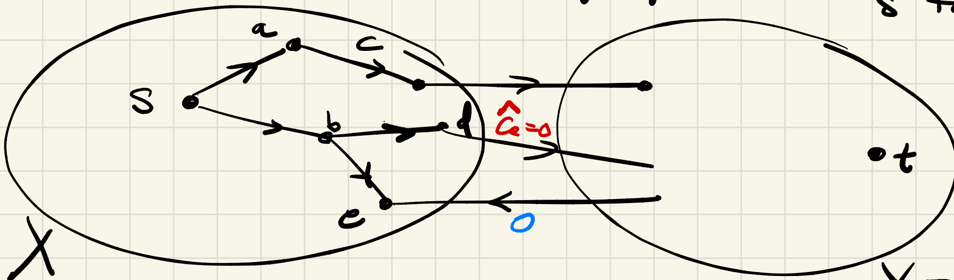


Ex s — a — b — c 1

s - b - a - t 1

Algorithm halts when no path from s to t in residual graph.

Final residual graph: no path from s to t.



X

$Y = V - X$

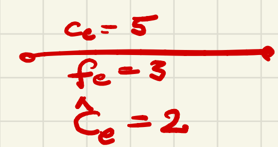
no path from s to t $\Rightarrow$

(a) forward edges from X to Y must have capacity 0. $\hat{c}_e = 0$ in residual graph. $\hat{c}_e = c_e - f_e$

(b) reverse edges from Y to X must have flow 0.

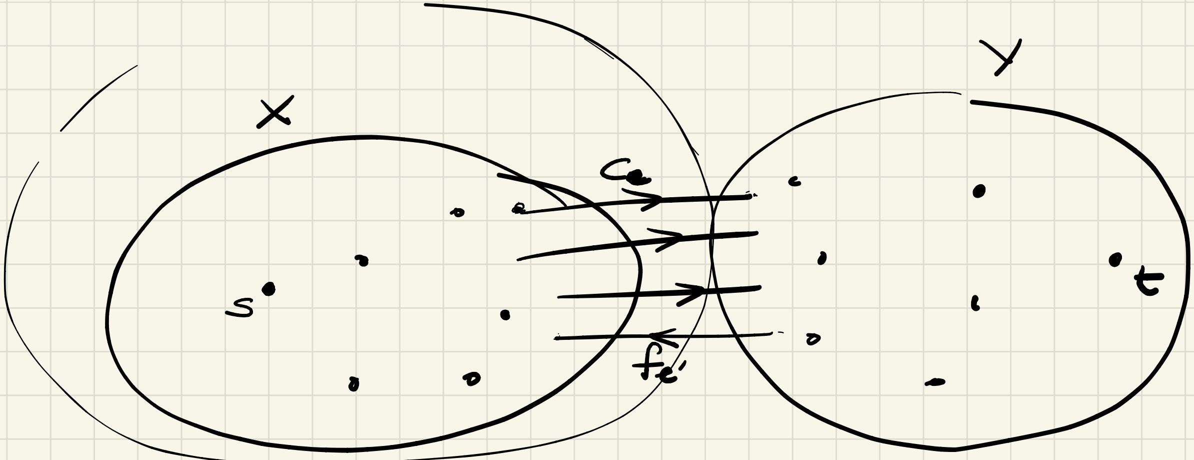
$$\text{Flow from } s \text{ to } t = \sum_{\substack{e=(u,v): \\ u \in X \,\&\, v \in Y}} c_e = cap(X,Y).$$

flow $= cap(X,Y)$ for some cut (X,Y).



$\therefore$ max flow.

$O(mn)$ flow paths $\Rightarrow$ $O(m^2 n)$ time



$$\frac{\text{maxflow} \quad \text{min cut}}{\text{flow} \;\; \leq \;\; \text{cut}}$$

$=$

$f_e = c_e$ for every edge from X to Y.

$f_{e'} = 0$ for every edge from Y to X.

Flow from X to Y = flow out of s = flow into t.

$$\sum_{e \text{ from } X \text{ to } Y} c_e = \mathrm{cap}(X, Y).$$

any flow $\leq \mathrm{cap}(X,Y)$ $\Big\}$ $\therefore$ max flow.