Zero sum games & Multiplicative weight update Algorithm.

2 person zero sum games:

Rock paper scissor:

$$\begin{array}{c c} & \text{Bob} \\ & \begin{array}{ccc} R & P & S \end{array} \\ \text{Alice} \begin{array}{c} \to R \\ \times P \\ \to S \end{array} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{array}$$

entry of matrix = Alice's payoff

= − Bob's payoff.

Strategy = instructions for someone else to play on your behalf.

$P[R] = P[P] = P(S) = 1/3$.

✓ $P[R] = P[S] = 1/2$.

Bob.

R: $1/2$

P: 0

S: $-1/2$

Bob

R: 0

P: 0

S: 0

even – odd game :

$$\begin{array}{c|c} & \begin{array}{cc} 0 & 1 \end{array} \\ x_0 \to 0 & \multirow{2}{*}{$\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}$} \\ x_1 \to 1 & \end{array}$$

if sum even then Alice's payoff = sum.

if odd then Alice's payoff = – sum

Alice: 0 wp x_0
1 wp x_1

$x_0 = 2/3$
$x_1 = 1/3$

Alices's payoff:

Bob $\begin{cases} 0: & x_0 \cdot 0 + x_1 \cdot (-1) = -x_1 \\ 1: & -x_0 + 2x_1 . \end{cases}$

$\min\{-x_1, -x_0 + 2x_1\}.$

$\min\{-1/3, -\frac{2}{3} + 2 \cdot \frac{1}{3}\} = -1/3.$

Alice: Pick x_0, x_1 to maximize $\min\{-x_1, -x_0 + 2x_1\}.$

<u>LP</u>

$$\begin{cases} \max z \\ \left.\begin{array}{l} z \le -x_1 \\ z \le -x_0 + 2x_1 \end{array}\right\} \\ x_0 + x_1 = 1 \\ x_0, x_1 \ge 0 \end{cases}$$

$z = \min\{-x_1, -x_0 + 2x_1\}$

$z \le \min\{-x_1, -x_0 + 2x_1\}$

<u>OPT</u>: $x_0 = 3/4$ $x_1 = 1/4.$

payoff = $-1/4$

$$A = \begin{array}{c c} & \begin{array}{cc} y_0 & y_1 \end{array} \\ \begin{array}{c} 0 \\ 1 \end{array} & \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix} \end{array}$$

Bob announces strategy y_0, y_1.

Alice's payoff:
$0: \quad -y_1$
$1: \quad -y_0 + 2y_1$

$$\max\{-y_1, -y_0 + 2y_1\}$$

Bob pick y_0, y_1 : $\max\{-y_1, -y_0+2y_1\}$ is minimized.

$$(x_1 \;\; x_2)\begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -x_2 & -x_1+2x_2 \end{pmatrix}$$

max Z.
$x^T A \geq z(1 \;\; 1)$
$[1 \;\; 1]\vec{x} = 1$
$\vec{x} \geq 0$

$$A = \begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} y_0 \\ y_1 \end{bmatrix} = \begin{bmatrix} -y_1 \\ -y_0+2y_1 \end{bmatrix}$$

min z.
$Ay \leq z\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
$[1 \;\; 1]\begin{bmatrix} y \end{bmatrix} = 1.$
$\vec{y} \geq 0$

min Z
$-y_1 \leq Z$
$-y_0 + 2y_1 \leq Z$
$y_0 + y_1 = 1$
$y_0, y_1 \geq 0.$

$y_0 = 3/4$
$y_1 = 1/4.$

payoff $= -1/4.$

duals.

$$\begin{pmatrix} \longleftarrow x^T \longrightarrow \end{pmatrix} \begin{pmatrix} & & \\ & A & \\ & & \end{pmatrix} \begin{pmatrix} \\ y \\ \\ \end{pmatrix} = x^T A y = \text{payoff}$$

when Alice plays x & Bob plays y.

von Neumann's min max thm:

$$\max_{x} \min_{y} x^T A y = \min_{y} \max_{x} x^T A y$$

Multiplicative weight update Algorithm (MWU).

n experts $\quad E_1, E_2, \ldots, E_n$

i^{th} expert on day t : loss $\ell_i^{(t)} \in [0,1]$ $\qquad T = \#$ days.

$L^* =$ total loss of best expert over T days.

Strategy: Initialize $w_i = 1 \quad i = 1$ to n. $\qquad W = \sum_{i=1}^{n} w_i$

Each day : Expert i w.p. $\dfrac{w_i}{W}$

end of day : expert j : loss $\ell_j^{(t)}$

$$w_j \leftarrow w_j (1-\varepsilon)^{\ell_j^{(t)}}$$

$L =$ expected loss suffered by MWU over T days.

Thm: $\quad L \le \dfrac{\ln n}{\varepsilon} + (1+\varepsilon) L^*$

Pf Sketch: $W(t) = \sum_{i=1}^{n} w_i^{(t)}$

$W(0) = n$

Best expert's weight on day T $= (1-\varepsilon)^{L^*}$

L^*

$$(1-\varepsilon)^{L^*} \leq W(T)$$

L_t = expected loss suffered by MWU on day t.

$$W(t+1) \leq W(t)(1-\varepsilon L_t)$$

$$(1-\varepsilon)^{L^*} \leq W(T) \leq \overset{W(0)}{\overset{\shortparallel}{n}} \prod_{t=1}^{T} (1-\varepsilon L_t)$$

$$L^* \ln(1-\varepsilon) \leq \ln n + \sum_{t=1}^{T} \ln(1-\varepsilon L_t).$$

$$\text{(ln } (1-\varepsilon) \geq -(\varepsilon+\varepsilon^2)\text{, and } \ln(1-\varepsilon L_t) \leq -\varepsilon L_t)$$

$$-L^*(\varepsilon+\varepsilon^2) \leq \ln n - \varepsilon \sum_t L_t \qquad \sum_t L_t = L$$

$$-L^*(1+\varepsilon) \leq \frac{\ln n}{\varepsilon} - L \Rightarrow L \leq \frac{\ln n}{\varepsilon} + (1+\varepsilon) L^*$$